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Automatic classification of normal forms

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Abstract

The aim of this paper is to demonstrate a specific application of Computer Algebra to bifurcation theory with symmetry. The classification of different bifurcation phenomena in case of several parameters is automated, based on a classification of Gröbner bases of possible tangent spaces. The computations are performed in new coordinates of fundamental invariants and fundamental equivariants, with the induced weighted ordering. In order to justify the approach the theory of intrinsic modules is applied. Results for the groups D_3, Z_2, and $Z_2 \times Z_2$ demonstrate that the algorithm works independent of the group and that new results are obtained.

Keywords. bifurcation theory, singularity theory, symmetry, systems with several parameters, Computer Algebra, Gröbner bases.

AMS subject classification. 13 P10, 34 A47, 34 C20, 58 E09

1 Introduction

We present a systematic and algorithmic approach for the theory of bifurcations with symmetry. In applications the solutions often depend on several parameters. By natural geometric arrangements symmetry comes in, i.e. a group action. It is well accepted that by Lyapunov Schmidt reduction it suffices to study low dimensional problems. Golubitsky, Stewart, Schaeffer [11, 12] have described the singularity theory for various groups in detail. Similarity of phenomena is identified by contact equivalence and classified by codimension. Golubitsky, Stewart, Schaeffer present several examples of normal forms.

Our aim is to study possible phenomena by algorithmic generation of representatives of contact equivalence classes. The emphasis is not to study problems of high codimension, but to develop an algorithm for arbitrary group actions.

The main tool is Computer Algebra, especially Gröbner Bases. Several people [1, 8, 2, 19] have used Computer Algebra in dynamical systems and bifurcation theory, but our approach is different from those procedures since our new classification of singularities is based on a classification of Gröbner bases of possible tangent spaces.

A list of representatives explains which phenomena may occur and is the basis for the development of numerical algorithms. The question whether two vector fields are in the same equivalence class is not relevant because one has to use numerical algorithms for applications anyway and for theoretical purposes it suffices to have one representative.

We make systematic use of Computer Algebra methods. This includes that instead of germs we handle polynomials which creates new problems but nevertheless it seems to be a reasonable approach.

The paper is organized as follows. Section 2 introduces bifurcation theory while section 3 recalls the basics of Gröbner bases for modules and parameter handling. A section on invariant theory of discrete groups and the subgroups of the contact equivalence completes the preparational work. The idea of our algorithm outlined in section 5 is explained in more detail and justified in the following sections. The example of D_3 demonstrates the algorithm in detail. Computed bifurcation problems for Z_2 and $Z_2 \times Z_2$ complete the paper.

2 Notions of Singularity Theory

In this section the notations from singularity theory are given which are needed to formulate the aim of this paper and the algorithm. Since one studies the phenomena locally, germs are used. Secondly, one may restrict to the case that the point of interest is the origin. We use the notations in [12] as far as possible.

Two C^∞ functions $f_i : U_i \to \mathbf{R}, i = 1, 2$, where U_1, U_2 are open neighborhoods of $0 \in \mathbf{R}^n$, are identified, if $f_{1|U_1 \cap U_2} = f_{2|U_1 \cap U_2}$. This is an equivalence relation and the equivalence classes are called *germs*. The spaces of germs in the variable λ or variables $x_1, \ldots, x_n, \lambda$ are denoted by ϵ_λ or $\epsilon_{x,\lambda}$, respectively. These spaces are rings. By $\vec{\epsilon}_{x,\lambda}$ map-germs of C^∞ vector fields $f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ are denoted and $\overset{\leftrightarrow}{\epsilon}_{x,\lambda}$ is the space of germs of matrix valued functions $f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^{n \times n}$. $\vec{\epsilon}_{x,\lambda}$ and $\overset{\leftrightarrow}{\epsilon}_{x,\lambda}$ are finitely generated modules over the ring $\epsilon_{x,\lambda}$.

Symmetry is described by a linear representation $\vartheta : G \to Gl(\mathbf{R}^n)$. In this paper we consider finite groups G only. By this fact we may assume that ϑ is *orthogonal* and of course we assume that G and $\vartheta(G)$ are isomorphic, i.e. ϑ is *faithful*.

Usually the notations Γ instead of $\vartheta(G)$ and $\gamma \in \Gamma$ instead of $\vartheta(t), t \in G$ are used, but we prefer to distinguish the abstract group G and their action because this is more precise and avoids misunderstandings.

A germ $f \in \epsilon_{x,\lambda}$ is called *invariant*, if

$$f(\vartheta(t)x, \lambda) = f(x, \lambda), \quad \forall t \in G, \tag{1}$$

and a map-germ $f \in \vec{\epsilon}_{x,\lambda}$ is called ϑ-*equivariant*, if

$$f(\vartheta(t)x, \lambda) = \vartheta(t) f(x, \lambda), \quad \forall t \in G. \tag{2}$$

The *commuting* matrix-germs $S \in \overset{\leftrightarrow}{\epsilon}_{x,\lambda}$ satisfy

$$S(\vartheta(t)x, \lambda) = \vartheta(t) S(x, \lambda) \vartheta(t^{-1}), \quad \forall t \in G. \tag{3}$$

We denote the spaces by $\epsilon_{x,\lambda}(\vartheta)$, $\vec{\epsilon}_{x,\lambda}(\vartheta)$ and $\overset{\leftrightarrow}{\epsilon}_{x,\lambda}(\vartheta)$.

We also need to classify the space $\mathcal{L}(\vartheta)$ of commuting matrices S in $\mathbf{R}^{n\times n}$

$$S = \vartheta(t) S \vartheta(t^{-1}), \quad \forall t \in G.$$

By $\mathcal{L}(\vartheta)^0$ one denotes the connected component $\mathcal{L}(\vartheta) \cap Gl(\mathbf{R}^n)$ which contains the identity.

Example 2.1 *For example for the representation of* $D_3 = \{id, r, r^2, s, sr, sr^2\}$

$$\vartheta(r) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \vartheta(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then

$$\mathcal{L}(\vartheta)^0 = \{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} | \quad a > 0, b > 0\}.$$

In order to study symmetric bifurcation problems one investigates germs $g \in \vec{\epsilon}_{x,\lambda}(\vartheta)$ with $g(0,0) = 0$ and $g_x(0,0) = 0 \in \mathbf{R}^{n\times n}$. We use the notations $\mathcal{M}_{x,\lambda}(\vartheta) = \{g \in \epsilon(\vartheta) | g(0,0) = 0\}$, $\vec{\mathcal{M}}_{x,\lambda}(\vartheta) = \{g \in \vec{\epsilon}(\vartheta) | g(0,0) = 0\}$.

Definition 2.2 *([12, p. 166]) Let ϑ be an orthogonal, faithful representation of a finite group G. Let $f_1, f_2 \in \vec{\mathcal{M}}_{x,\lambda}(\vartheta)$ be two map-germs with $d_x f_i(0,0) = 0, i = 1, 2$. f_1 and f_2 are* ϑ-contact equivalent *($f_1 \sim f_2$), if there exists an invertible change of coordinates $(x, \lambda) \to (X(x, \lambda), \Lambda(\lambda))$ with $X \in \vec{\mathcal{M}}_{x,\lambda}(\vartheta), \Lambda \in \mathcal{M}_\lambda, S \in \overset{\leftrightarrow}{\epsilon}_{x,\lambda}(\vartheta)$ with*

$$f_1(x, \lambda) = S(x, \lambda) f_2(X(x, \lambda), \Lambda)) \tag{4}$$

and

$$S(0,0), d_x X(0,0) \in \mathcal{L}(\vartheta)^0. \tag{5}$$

If $\Lambda(\lambda) \equiv \lambda$ then f_1 and f_2 are called strongly ϑ-contact equivalent*, denoted by $f_1 \sim_s f_2$.*

A representative of the equivalence relation is called *normal form*. This notion of normal form should not to be confused with the normal forms in the context of Gröbner bases (section 3). The inequalities (5) are necessary to preserve stability assignments. It is no restriction to allow only linear changes of coordinates in the range, by a result due to Mather [16].

The investigation of equivalence classes of germs is done by considering the linearization. The tangent space at the orbit of a germ is the key for the understanding of symmetric bifurcation problems.

Definition 2.3 *([12, p. 167]) Let $g \in \vec{\mathcal{M}}_{x,\lambda}(\vartheta)$ with $g_x(0,0) = 0$. The* ϑ-equivariant restricted tangent space *is defined by*

$$\mathcal{RT}(g,\vartheta) = \{f \in \vec{\epsilon}_{x,\lambda}(\vartheta) | \exists X \in \vec{\mathcal{M}}_{x,\lambda}(\vartheta), S \in \overset{\leftrightarrow}{\epsilon}_{x,\lambda}(\vartheta) \text{ with } f = S(x,\lambda)g(X(x,\lambda),\lambda)\} \quad (6)$$

Note that $\mathcal{RT}(g,\vartheta)$ is a $\epsilon_{x,\lambda}$-module and a submodule of $\vec{\epsilon}_{x,\lambda}$. For many group actions $\vec{\mathcal{M}}_{x,\lambda}(\vartheta) = \vec{\epsilon}_{x,\lambda}(\vartheta)$ holds, but if ϑ contains the trivial irreducible representation then $\vec{\mathcal{M}}_{x,\lambda}(\vartheta)$ is a proper subspace of $\vec{\epsilon}_{x,\lambda}(\vartheta)$ as an $\mathbf{R}$-vector space, i.e.

$$\vec{\epsilon}_{x,\lambda}(\vartheta) = \vec{\mathcal{M}}_{x,\lambda}(\vartheta) + \mathbf{R}\{Y_1, \ldots, Y_m\}.$$

Definition 2.4 *Let $g \in \vec{\mathcal{M}}_{x,\lambda}(\vartheta)$ with $g_x(0,0) = 0$. The* ϑ-equivariant tangent space *is defined by*

$$\mathcal{T}(g,\vartheta) = \mathcal{RT}(g,\vartheta) + \mathbf{R}\{g_x Y_1, \ldots, g_x Y_m\} + \epsilon_\lambda\{g_\lambda\}. \quad (7)$$

$\mathcal{T}(g,\lambda)$ is a $\mathbf{R}$-vector space, but no $\epsilon_{x,\lambda}$-module like the restricted tangent space. If ϑ contains no trivial part ($\vec{M}_{x,\lambda}(\vartheta) = \vec{\epsilon}_{x,\lambda}(\vartheta)$), then $\mathcal{T}(g,\vartheta)$ is a ϵ_λ-module. Throughout this paper we restrict to the case that ϑ contains no trivial irreducible representation.

While the restricted tangent space refers to the notion of strong equivalence, the tangent space corresponds to contact equivalence itself.

The main application of singularity theory to bifurcation theory is the study of *unfoldings*. For given $g(x,\lambda)$ one considers $\tilde{g} : \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^k \to \mathbf{R}^n$, $(x,\lambda,\alpha) \to \tilde{g}(x,\lambda,\alpha_1,\ldots,\alpha_k)$ and the bifurcation diagram of $\tilde{g}(x,\lambda,\alpha_1,\ldots,\alpha_k)_{\alpha=\alpha_0}$ for a fixed value of α. Loosely speaking, an *universal* unfolding of g is a $\tilde{g}$ which covers all deformations of the bifurcation diagram for g, but has minimal number of parameters α_i.

This universal unfolding can be obtained using the tangent space. As a $\mathbf{R}$-vector space $\mathcal{T}(g,\vartheta)$ has a complementary space spanned by $v_1,\ldots,v_l$ provided $\mathcal{T}(g,\vartheta)$ has finite codimension. Then $\tilde{g}(x,\lambda,\alpha) = g(x,\lambda) + \sum_{j=1}^{l} \alpha_j v_j$ is a universal unfolding of g, see [12]. The number of parameters (the codimension of $\mathcal{T}(g,\vartheta)$) is the *codimension* of g.

Since the restricted tangent space is easier to handle than the tangent space itself, it is interesting to know that $\mathcal{T}(g,\vartheta)$ has finite codimension if and only if $\mathcal{RT}(g,\vartheta)$ has finite codimension. For $G = \{id\}$ and one equation ($n = 1$) this is formulated in [11, p. 127] while the general case is mentioned in [12] and goes back to an unpublished proof by J. Damon [7].

The first task is to produce for various group actions lists of representatives of different ϑ-contact equivalence classes up to a certain small codimension. Based on these lists numerical analysts may develop algorithms for the numerical treatment of bifurcation problems with symmetry and with several parameters. With these numerical algorithms it can be checked whether two functions are in the same contact equivalence class.

Our aim is the automation of the theoretical process. In the following sections we describe how to use Computer Algebra to generate these lists in a systematic way.

3 The Main Tool: Gröbner Bases

In this section some basic facts about Gröbner bases will be recalled. They are needed in the following. The Buchberger Algorithm for the computation of Gröbner bases may be viewed as generalization of the Euclidian Algorithm or the Gaussian elimination for the solution of systems of linear equations.

The polynomial ring $K[x_1,\ldots,x_n]$ over the field K (usually $\mathbf{Q}$ or some algebraic field extension of $\mathbf{Q}$) is generated by *monomials* or *terms* $x_1^{i_1} \cdots x_n^{i_n}$ which are ordered. In order

to fit with computations the ordering $<_T$ has to be *admissible*: if $m_1 <_T m_2$ for some monomials m_1, m_2 then for all monomials m it is required that $m \cdot m_1 <_T m \cdot m_2$.
Examples: Let $x_1 \gg x_2 \gg \cdot \gg x_n$ be the ordering among the variables. The *lexicographical* ordering is defined by

$$x_1^{i_1} \cdots x_n^{i_n} \leq_l x_1^{j_1} \cdots x_n^{j_n},$$

if there exist $1 \leq \nu \leq n$ with $i_\mu = j_\mu$ for $1 \leq \mu \leq \nu - 1$ and $i_\nu < j_\nu$.
For the *weighted ordering* one uses a vector $(d_1, \dots, d_n) \in \mathbf{N}^n$.

$$x_1^{i_1} \cdots x_n^{i_n} \leq_w x_1^{j_1} \cdots x_n^{j_n}, \text{ if}$$

$$\sum_{\nu=1}^{n} i_\mu d_\nu < \sum_{\nu=1}^{n} j_\nu d_\nu \textbf{ or } \left(\sum_{\nu=1}^{n} i_\mu d_\nu = \sum_{\nu=1}^{n} j_\nu d_\nu \textbf{ and } x_1^{i_1} \cdots x_n^{i_n} \leq_l x_1^{j_1} \cdots x_n^{j_n} \right).$$

Each vector $d \in \mathbf{N}^n$ corresponds to a different grading of the polynomial ring. Compare these orderings with those in [3].

Given an ordering one denotes for a polynomial $f \in K[x]$ the *leading term* by $lt(f)$, i.e.

$$f = a \cdot lt(f) + \sum_{m <_T lt(f)} a_m \cdot m, \quad a, a_m \in K, a \neq 0. \tag{8}$$

We shall write $LT(F) = \{lt(f_1), \dots, lt(f_m)\}$ for the leading terms of a set of polynomials $F = \{f_1, \dots, f_m\}$. By $T(f)$ we denote the set of terms in the representation (8) of f.

Loosely speaking, a polynomial f_1 is simpler than f_2, if $lt(f_1) < lt(f_2)$. Polynomials are made 'simpler' by an elementary operation called *reduction*. Given some polynomials $F = \{f_1, \dots, f_m\} \subset K[x]$ and $f \in K[x]$ one writes $f \to_F g$, if there exists a $t \in T(f)$, a polynomial $f_i \in F$, and a constant $a \neq 0$ such that $t = m \cdot lt(f_i)$ and

$$f = a \cdot m \cdot f_i + g \text{ with } t \notin T(g). \tag{9}$$

If $t = lt(f)$ the reduction $\to_F$ is called a *top-reduction* and we have $lt(g) <_T lt(f)$ or $g = 0$. The notation $\to_F^*$ is more general. It includes $f \to_F^* f$ and $f \to_F g_1 \to_F g_2 \cdots \to_F g_r$ as $f \to_F^* g_r$. In other words it is the transitive and reflexive closure of $\to_F$. The operation $\to_F$ is easily implemented. A polynomial that can't be further reduced is called a *normal form* modulo F. Observe that this normal form is something completely different than the normal form defined in section 2 and should not be confused.

Definition 3.1 *([3]) A set $F = \{f_1, \dots, f_m\} \subset K[x] \setminus \{0\}$ is a* Gröbner basis *of the ideal $\mathcal{A} = (f_1, \dots, f_m)$ (wrt $\leq_T$) if it satisfies $f \to_F^* 0$ for all polynomials f in the ideal $\mathcal{A}$ which is generated by F.*

There are two important facts about Gröbner bases.
Consider the ideal generated by $f_1, \dots, f_m$. By abuse of language the computation of a Gröbner basis $\{g_1, \dots, g_l\}$ of this ideal wrt the lexicographical ordering is called the *solution* of the system $f_1(x) = 0, \dots, f_m(x) = 0$. This is justified by the fact that the Gröbner basis often is of triangular form $g_1(x_1, \dots, x_n), g_2(x_2, \dots, x_n), \dots, g_n(x_n)$, which is easily solved by backward substitution. In [15] it is shown that each radical of an ideal is the intersection of maximal ideals which are generated by polynomials in triangular form.

Secondly, for every polynomial f and every ideal $\mathcal{A}$ one can immediately decide algorithmically whether $f \in \mathcal{A}$ or not (ideal membership test).
We will use this in the following way: Assume $f(x, p) \in K[p][x]$ depends on some parameters p and one is interested in the values $p_0 \in \bar{K}$ such that $f(x, p_0)$ is the element

of an ideal $\mathcal{A} \subset \bar{K}[x]$. Demanding $f(x,p) \rightarrow_F^* 0$ where F is a Gröbner basis of $\mathcal{A}$ gives conditions on the parameters p. Since F is a Gröbner basis the normal form of f wrt F is unique. So the conditions are given by the coefficients of the normal form.

Of course the computation of a Gröbner basis can't be based on the Definition 3.1 since the formulation involves all polynomials in an ideal. But the conditions are generated by a finite number of conditions, the reduction of the *S-polynomials*. Let $f_i(x) = a^i m_i + \sum_{n<m_i} a^i_m m, i = 1,2$ then

$$S(f_1, f_2) = a_2 \frac{\operatorname{lcm}(m_1, m_2)}{m_1} f_1 - a_1 \frac{\operatorname{lcm}(m_1, m_2)}{m_2} f_2$$

is called the S-polynomial of f_1 and f_2, where lcm is the abbreviation of least common multiple. Observe that by subtraction the leading terms of the two summands cancel and thus a reduction modulo $\{f_1, f_2\}$ is *not* obvious.

Theorem 3.2 *([3, Thm.5.48 p. 211]) Let $F = \{f_1, \ldots, f_m\} \subset K[x] \setminus \{0\}$ and let $\mathcal{A}$ be the ideal generated by this set. The following statements are equivalent.*

i) F is a Gröbner basis of $\mathcal{A}$.

ii) $S(f_i, f_j) \rightarrow_F^* 0 \quad 1 \leq i < j \leq m$.

Theorem 3.2 is the key to the Buchberger algorithm. Starting with any set F the S-polynomials are reduced and non-zero normal forms are added to the set F. But there are a lot of details in the Buchberger algorithm and the computation time depends sensitively on these details.

The concept of Gröbner bases extends naturally to *modules*. Let $b_1, \ldots, b_r$ be new variables which represent a module basis. Then one computes in the free module

$$H_1(K[x,b]) = \{\sum_{j=1}^{r} h_j(x) b_j | h_j \in K[x]\}.$$

For a set $F \subset H_1(K[x,b])$ the reduction $\rightarrow_F$ is defined similarly to (9), but $m \in K[x]$. Secondly, S-polynomials are defined only for $f_i, f_j \in F$ if there exists a b_k with $b_k | lt(f_i)$ and $b_k | lt(f_j)$. By these modifications the variant of the Buchberger algorithm computes a Gröbner basis that now generates a submodule. A formal description based on graded Gröbner bases can be found in [3].

Definition 3.3 *([3]) Let $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$ be homogeous with respect to a grading Γ. A d_1-d_2-Gröbner basis is a Gröbner basis where only S-polynomials with*

$$d_1 \leq \Gamma(S(f_i, f_j)) \leq d_2$$

are considered.

For $H_1(K[x,b])$ choose the grading Γ with $\Gamma(x_i) = 0, \Gamma(b_j) = 1$. Then elements in $H_1(K[x,b])$ are homogeneous of degree 1 and a 1-1-Gröbner basis is a Gröbner basis of a module.

We will need to determine whether a submodule has *finite codimension*. For ideals the codimension is checked with the Hilbert function that is implemented in common Computer Algebra systems. The value of the Hilbert function is either a polynomial (if the codimension is infinite) or the value of the codimension. For modules generated by $f_1, \ldots, f_m$ one considers the leading terms $b_i m_{j_i}, i = 1, \ldots, r, j_i = 1, \ldots, r_i$, where m_{j_i} are monomials in $K[x]$. For $i = 1, \ldots, r$ one considers the codimension of the ideals generated by $m_{j_i}, j_i = 1, \ldots, r_i$. If all these ideals have finite codimension then the module has finite codimension, but in general the codimension is not the sum of the codimensions of the ideals. If a Gröbner basis of the submodule is known the calculation of the finite codimension is easier. It is the number of monomials which are not elements of the submodule generated by the leading terms.

Finally we discuss the role of parameters in the computation of Gröbner bases. The computation of Gröbner bases depending on parameters is a hard problem and should be avoided, if possible. There are two different concepts for the treatment of parameters. In the approach in [21] comprehensive Gröbner bases are introduced that are Gröbner bases for each value of the parameters. The concept in the Gröbner Package [17] in REDUCE is different from that. The computed polynomials form a Gröbner basis for almost all values of the parameters. The exceptions are stored and available on request.

4 Groups, Discrete and Continuous

The computation of singularities depends heavily on the use of new coordinates, the invariants and equivariants of the representation $\vartheta : G \to Gl(\mathbf{R}^n)$. So in this section we recall the induced representation of ϑ on polynomial mappings. The second point is the interpretation of the equivalence classes as orbits of a Lie group.

Since Hilbert it is known that the ring of invariant polynomials denoted by $\mathbf{R}[x]_\vartheta$ is generated by a finite number of invariants called the *Hilbert basis*. In commutative algebra one improves the basis such that n invariants $\sigma_1, \ldots, \sigma_n$ form the *primary invariants* and the *secondary invariants* $\eta_1, \ldots, \eta_s$ (with $\eta_1 = 1$) form the basis of $\mathbf{R}[x]_\vartheta$ over $\mathbf{R}[\sigma_1, \ldots, \sigma_n]$ as a *free* module. This means that every $p \in \mathbf{R}[x]_\vartheta$ has a *unique* representation

$$p(x) = \sum_{i=1}^{s} \eta_i(x) p_i(\sigma_1(x), \ldots, \sigma_n(x)), \quad p_i \in \mathbf{R}[\sigma_1, \ldots, \sigma_n].$$

Details of invariant theory can be found e.g. [20].

The ϑ-equivariant polynomial mappings form a module $\mathbf{R}[x]_\vartheta^\vartheta$ over the invariant ring which is finitely generated. Moreover the module is free over the subring $\mathbf{R}[\sigma_1, \ldots, \sigma_n]$, see e.g. [22],[10]. If this basis is denoted by $b_1, \ldots, b_r$ it means that every equivariant $f(x)$ has a *unique* representation

$$f(x) = \sum_{i=1}^{r} b_i(x) p_i(\sigma_1(x), \ldots, \sigma_n(x)) \quad \text{with}$$

some polynomials $p_i \in \mathbf{R}[x]$. The same facts are valid for the module of commuting matrices. In order to handle the restricted tangent space we will need $\vec{M}_x(\vartheta) = \{f \in \mathbf{R}[x]_\vartheta^\vartheta \,|\, f(0) = 0\}$. Since we assume that ϑ contains no trivial irreducible representation in its decomposition into irreducible representations, we have $\vec{M}_x(\vartheta) = \mathbf{R}[x]_\vartheta^\vartheta$.

Example 4.1 *(Example 2.1 continued) For the representation ϑ in example 2.1 the primary invariants are given by $\sigma_1(x) = x_1^2 + x_2^2$, $\sigma_2(x) = x_3^2$, $\sigma_3(x) = -x_1^3 + 3 \cdot x_1 x_2^2$ and the secondary invariants are $\eta_1(x) = 1, \eta_2(x) = -3 \cdot x_1^2 \cdot x_2 \cdot x_3 + x_2^3 \cdot x_3$. The fundamental equivariants are*

$$\begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2 \cdot x_3 \\ -x_1 \cdot x_3 \\ 0 \end{pmatrix}, \begin{pmatrix} -x_1^2 + x_2^2 \\ x_1 \cdot x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} -x_1 \cdot x_2 \cdot x_3 \\ -x_1^2 \cdot x_3 + x_2^2 \cdot x_3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -3 \cdot x_1^2 \cdot x_2 + x_2^3 \end{pmatrix}.$$

Since ϑ decomposes as the sum of two nontrivial irreducible representations the equivariants generate $\vec{M}_x(\vartheta)$ as well. The number of fundamental commuting matrices is 18.

Part of the results above are relevant for germs as well. Due to Schwarz (see [12, Thm. 4.3 p. 46]) a Hilbert basis of $\mathbf{R}[x]_\vartheta$ generates as well the invariant germs $\epsilon_x(\vartheta)$. A second theorem was proved by Poénaru (see [12, Thm. 5.3 p. 51]). A set of polynomial mappings $b_1, \dots, b_r$ that generates $\mathbf{R}[x]_\vartheta^\vartheta$ over $\mathbf{R}[x]_\vartheta$ also generate $\vec{\epsilon}_x(\vartheta)$ over $\epsilon_x(\vartheta)$.

If $S_1, \dots, S_l$ generate the module of commuting matrices in $\mathbf{R}[x]^{n\times n}$ then they generate $\overset{\leftrightarrow}{\epsilon}_x(\vartheta)$ as well.

These facts allow a rewriting of the restricted tangent space: $\mathcal{RT}(g,\vartheta)$ is the submodule generated by $S_1 g, \dots, S_l g, g_x b_1, \dots, g_x b_r$, where b_i are the generators of $\vec{M}_x(\vartheta)$, see [12, p. 169].

We are not aware of investigations concerning the further structure of primary and secondary invariants and the fact that $\mathbf{R}_\vartheta^\vartheta$ is a free $\mathbf{R}[\sigma]$-module inherit to the germs. Nevertheless the theorems by Schwarz and Poénaru justify the computation with polynomials instead of germs. In contrast to the usual approach we use that $\mathbf{R}[x]_\vartheta$ and $\mathbf{R}[x]_\vartheta^\vartheta$ are free modules over $\mathbf{R}[\sigma_1, \dots, \sigma_n]$, since the representations are unique.

We use new variables $\sigma_1, \dots, \sigma_n, \eta_2, \dots, \eta_s, b_1, \dots, b_r$ and compute in $\mathbf{R}[\sigma] + \oplus_{i=2}^s \eta_i \mathbf{R}[\sigma] \subset \mathbf{R}[\eta, \sigma]$ and $H_1(K[\sigma, b])$. The advantages are lower complexity of polynomials and avoiding the computation with tupels.

The grading of $\mathbf{R}[x]_\vartheta$ inherits a weighted ordering in $\mathbf{R}[\eta,\sigma]$ in a natural way: $\eta_s \gg \eta_{s-1} \gg \cdots \gg \eta_2 \gg \sigma_n \gg \cdots \gg \sigma_1$ and the tupel of weights $(e_s, \dots, e_1, d_n, \dots, d_1)$ where e_i and d_i are the degrees of $p_i(x)$ and $\sigma_i(x)$, respectively. Analogously, the weighted ordering for $\mathbf{R}[b,\sigma]$ is defined by $b_r \gg \cdots \gg b_1 \gg \sigma_n \cdots \gg \sigma_1$ and weights $(e_r, \dots, e_1, d_n, \dots, d_1)$ where e_i, d_i are the degrees of $b_i(x)$ and $\sigma_i(x)$, respectively. In the following we will compute using this weighted ordering.

In this context we have to solve 4 problems:

(i) the computation of primary invariants σ_i and secondary invariants η_j

(ii) a.) the computation of fundamental equivariants b_i

b.) the computation of fundamental commuting matrices S_i

(iii) compute the polynomials p_i in the representation of an equivariant $f(x) \in \mathbf{R}[x]_\vartheta^\vartheta$ as

$$f(x) = \sum_{i=1}^{r} p_i(\sigma_1(x), \dots, \sigma_n(x)) \cdot b_i.$$

The first task is described in [20]. There is even a Maple package called Invar [14] available. Algorithms for the second and third task are presented in [10].

So far we have discussed the effect of the finite symmetry group, now we look at the group of equivalences of bifurcation problems.

Definition 4.2 *Assume $\vartheta : G \to Gl(\mathbf{R}^n)$ is an orthogonal, faithful linear representation and recall $\vec{\mathcal{M}}_{x,\lambda}(\vartheta) \subset \vec{\epsilon}_{x,\lambda}(\vartheta)$, $\mathcal{L}(\vartheta)^0, \mathcal{M}_\lambda \subset \epsilon_\lambda$ and $\overset{\leftrightarrow}{\epsilon}_{x,\lambda}(\vartheta)$ as defined in section 2. Then*

$$\begin{aligned} \mathcal{B} := \{(S,X,\Lambda) \in \; & (\overset{\leftrightarrow}{\epsilon}_{x,\lambda}(\vartheta), \vec{\mathcal{M}}_{x,\lambda}(\vartheta), \mathcal{M}_\lambda) | \\ & (X,\Lambda) \textit{ invertible }, S(0,0), d_x X(0,0) \in \mathcal{L}(\vartheta)^0\}, \end{aligned} \tag{10}$$

is the contact equivalence group *where the group composition is defined by*

$$(S_1, X_1, \Lambda_1) \circ (S_2, X_2, \Lambda_2) = (S_1 \cdot S_2 \circ (X_1, \Lambda_1), X_1 \circ (X_2, \Lambda_2), \Lambda_1 \circ \Lambda_2).$$

As already indicated $\mathcal{B}$ is a group. The orbit

$$\mathcal{O}_g = \{f \in \vec{\epsilon}_{x,\lambda}(\vartheta) | \exists (S, X, \Lambda) \in \mathcal{B} : f = (S, X, \Lambda) g\},$$

is the ϑ-contact equivalence class of g.

Strong equivalence corresponds to the subgroup

$$\mathcal{B}_s := \{(S, X, \Lambda) \in \mathcal{B} | \Lambda(\lambda) \equiv \lambda\}.$$

Other important subgroups are

$$\begin{array}{rcll} \mathcal{B}_\Lambda & = & \{(S,X,\Lambda) \in \mathcal{B}| & X(x,\lambda) \equiv x, S(x,\lambda) \equiv Id\}, \\ \mathcal{B}_{\mathcal{L}^0} & = & \{(S,X,\Lambda) \in \mathcal{B}| & X(x,\lambda) \equiv x, \Lambda(\lambda) \equiv \lambda, S(x,\lambda) \equiv S \in \mathcal{L}(\vartheta)^0\}, \\ \mathcal{B}_\lambda & = & \{(S,X,\Lambda) \in \mathcal{B}| & X(x,\lambda) \equiv x, \Lambda(\lambda) = \gamma \cdot \lambda, \gamma \in \mathbf{R}, \gamma > 0, S(x,\lambda) \equiv Id\}, \\ \mathcal{B}_x & = & \{(S,X,\Lambda) \in \mathcal{B}| & X(x,\lambda) = \sum_{i=1}^r \gamma_i b_i(x), \gamma_i \in \mathbf{R}, \gamma_i > 0, \\ & & & \Lambda(\lambda) \equiv \lambda, S(x,\lambda) \equiv Id\}, \end{array} \tag{11}$$

where b_i are the generators of $\vec{M}_x(\vartheta)$, which are linear and fulfill $d_x b_i(0,0) \in \mathcal{L}(\vartheta)^0$.

We will use these subgroups in section 9 to eliminate higher order terms ($\mathcal{B}_\Lambda$) and to restrict to a special representative of an orbit $\mathcal{O}_g$ ($\mathcal{B}_{\mathcal{L}^0}, \mathcal{B}_\lambda, \mathcal{B}_x$).

5 The Concept

In this section we present the basic idea of this paper. The further details are worked out in sections 8 and 9.

In computations it is not possible to work with germs or with polynomials of arbitrary degree, therefore we choose a finite-dimensional vector space $A \subset H_1(K[(\sigma_1, \dots, \sigma_n, \lambda), b])$ that is generated by some monomials. For example A corresponds to the space of polynomial mappings up to degree k. We will only compute singularities in A. That means that we make an Ansatz

$$g = \sum_{m \in A} c_m \cdot m, \tag{12}$$

where the monomials m generate A and $c_m \in K$ are unknowns that have to be determined. By abuse of language we use A simultaneously as vector space and as the set of monomials that form a vector space basis of A.

The first conditions are that $g(0) = 0$ and $g_x(0) = 0$ in order for g to be a singularity. For $g \in A$ an analogue of the tangent space for germs is defined.

Definition 5.1 *For $g \in A$ the restricted tangent space $RT(g, \vartheta) \subset H_1(K[(\sigma, \lambda), b])$ is defined to be the $K[\sigma, \lambda]$-module generated by $S_i g, i = 1, \dots, l, d_x g b_j, j = 1, \dots, r$ where the S_i form a module basis of the commuting matrices $K[x]^{n \times n}$ over $K[\sigma]$ and the b_j form a module basis of $\vec{M}_{x,\lambda}(\vartheta)$ over $K[\sigma]$. The tangent space $T(g, \vartheta)$ is defined as the $K[\lambda]$–module given by*

$$RT(g, \lambda) + K[\lambda]\{g_\lambda\}.$$

Here we use that ϑ contains no trivial irreducible representation.
Since the algebraic structure of $T(g, \vartheta)$ is very important we give another definition.

Definition 5.2 *A $K[\lambda]$-module $t \in H_1(K[(\sigma, \lambda), b])$ is called* a possible tangent space *if there exists a subspace rt which is also a $K[\sigma, \lambda]$-module and a $q \in H_1(K[(\sigma, \lambda), b])$ such that $t = rt + K[\lambda]\{q\}$.*

The concept of reduction and Gröbner basis extends to this type of spaces.

Definition 5.3 *Let F be a set of polynomials in $H_1(K[(\sigma,\lambda),b])$ generating a submodule and let $q \in H_1(K[(\sigma,\lambda),b])$. Consider the reduction $\rightarrow_F^*$ with respect to an ordering in $H_1(K[(\sigma,\lambda),b])$ and the reduction $\rightarrow_q^*$ in $H_1(K[\lambda,u])$ with respect to the same order, where u stands for all monomials in $H_1(K[\sigma,b])$. We denote by $\rightarrow_{F,q}^+$ the transitive closure of the union of these two relations.*

The key idea is to classify the possible tangent spaces up to a certain codimension. This is based on a guess of $LT(rt)$ of leading terms of RT and one additional monomial r in $K[\sigma,\lambda]$ for generating the rest of t.

Then parameters $p(lt,m)$ are introduced in the following way: For each $lt \in LT(rt)$

$$P(b,\sigma,\lambda) = lt + \sum_{m<_T lt} p(lt,m)\cdot m, \quad p(lt,m) \in K, \text{ and} \tag{13}$$

$$Q(b,\sigma,\lambda) = r + \sum_{m<_T lt} p(r,m)\cdot m, \quad p(r,m) \in K. \tag{14}$$

Here $<_T$ denotes the weighted ordering introduced in section 4.

The assumption that the polynomials (13) form a Gröbner basis of rt in the variables σ_j, λ and module variables b_i gives conditions for the free parameters $p(lt,m) \in K$. That the polynomials in (13) and (14) form a Gröbner basis wrt the variable λ and the module variables σ_j, b_i gives conditions for the free parameters $p(r,m)$.

Conditions for the parameters c_m in the singularity $g \in A$ are now given by requiring

$$RT(g,\vartheta) \subset rt \text{ and } q \in t. \tag{15}$$

A second alternative is to demand

$$T(g,\vartheta) \subset t. \tag{16}$$

This corresponds to the notions of contact equivalence and strong contact equivalence.

After solving these conditions there may be still free parameters in g. They can be determined by choosing special representatives of the orbits of g under the groups $\mathcal{B}_\lambda, \mathcal{B}_x, \mathcal{B}_\Lambda, \mathcal{B}_{\mathcal{L}^0}$.

The ideas of this section are worked out in more detail in sections 8 and 9.

6 Intrinsic Modules and Determinacy

The concept of intrinsic ideals and submodules proves to be very important both for the theoretical foundation of our work as well as for some practical aspects. In a situation with one state variable, one parameter, and no symmetry present, intrinsic ideals have a particular nice form and can be used to give more precise results as in the equivariant context. However, as far as we know, this has never be explored in a systematic way. The fundamental ideas go back to the paper by Bruce, du Plessis and Wall [4] for general singularity theory. Gaffney [9] gives a translation of these results to bifurcation theory. Some more background material can be found in the monographs by Golubitsky and Schaeffer [11], Golubitsky, Stewart and Schaeffer [12] respectively. Let us briefly recall the definition. As before $\mathcal{B}$ denotes the group of transformations given in Definition 2.2.

Definition 6.1 *An ideal I in $\epsilon_{x,\lambda}(\vartheta)$ is called intrinsic, if it is invariant under the action of $\mathcal{B}$. Similarly a submodule M of $\vec{\epsilon}_{x,\lambda}(\vartheta)$ is intrinsic if it is invariant under the action of $\mathcal{B}$, i.e. $\forall\, f \in M, (S,X,\Lambda) \in \mathcal{B} :\ (S,X,\Lambda)f \in M$.*

In general $\mathcal{RT}(f,\vartheta)$ is not invariant under $\mathcal{B}$, but it has an intrinsic part.

Lemma 6.2 *For $f \in \vec{\epsilon}_{x,\lambda}(\vartheta)$ the set*

$$\mathcal{P}(f) = \{h \in \vec{\epsilon}_{x,\lambda}(\vartheta) \mid g \sim f \Rightarrow g+h \sim f\}, \tag{17}$$

is an intrinsic submodule (ideal if $f \in \vec{\epsilon}_{x,\lambda}(\vartheta)$ is replaced by $\epsilon_{x,\lambda}(\vartheta)$).

Observe that the set $\mathcal{P}(f)$ is independent of the representative of $\mathcal{O}_f$. So we may write as well $\mathcal{P}(\mathcal{O},\vartheta) = \{h \in \vec{\epsilon}_{x,\lambda}(\vartheta) \mid g \in \mathcal{O} \Rightarrow g+h \in \mathcal{O}\}$.

Definition 6.3 *$\mathcal{P}(f)$ is called the* intrinsic module *of f.*

We recall that a bifurcation problem f has finite codimension, if and only if the restricted tangent space has finite codimension, see also the remark in section 2. The codimension of f is given by the codimension of $\mathcal{T}(f,\vartheta)$. $\mathcal{P}(f)$ can be characterized as Lf, where $L \subset L\mathcal{B}$ is a maximal nilpotent Lie-algebra (Gaffney [9]). Therefore $\mathcal{P}(f) \subset \mathcal{RT}(f,\vartheta)$ and we have the trivial part of the following lemma.

Lemma 6.4 *For $f \in \vec{\epsilon}_{x,\lambda}(\vartheta)$*

$$\mathrm{codim}(\mathcal{RT}(f,\vartheta)) < \infty \Leftrightarrow \mathrm{codim}(\mathcal{P}(f)) < \infty. \tag{18}$$

Proof: We have seen that the trivial case follows from Gaffney's characterization of $\mathcal{P}(f)$ immediately. The other direction follows from Theorem 1.3 in Gaffney [9]. $\square$

Since $\mathcal{P}(f) \subset \mathcal{RT}(g,\vartheta)$ for all $g \in \mathcal{O}_f$ it is clear that

$$\mathcal{P}(f) \subseteq \bigcap_{g \in \mathcal{O}_f} \mathcal{RT}(g,\vartheta).$$

For the rest of this section let $f \in \vec{\epsilon}_{x,\lambda}(\vartheta)$ be a bifurcation problem of finite codimension. In view of Lemma 6.4 one could in principle do the computations *mod* $\mathcal{P}(f)$ if one would know $\mathcal{P}(f)$. Then f itself can be considered mod $\mathcal{P}(f)$. Secondly, due to Lemma 6.4 we can choose a complement V of the $\mathbf{R}$-vector space $\mathcal{P}(f)$ in $\vec{\epsilon}_{x,\lambda}(\vartheta)$ of finite dimension. Then

$$\begin{array}{rcll} \vec{\epsilon}_{x,\lambda}(\vartheta) & = & V+ & \mathcal{P}(f) \\ \forall g \in \mathcal{O}_f \quad \mathcal{RT}(g,\vartheta) & = & V_g+ & \mathcal{P}(g), \end{array}$$

where $V_g := \mathcal{RT}(g,\vartheta) \cap V$ and $\mathcal{P}(f) = \mathcal{P}(g)$. The action of $\mathcal{B}$ extends to an action of the set of all $V_g, g \in \mathcal{O}_f$. The element (S,X,Λ) maps g to $(S,X,\Lambda)g$ and V_g to $V_{(S,X,\Lambda)g}$. Thus one computes on $\mathcal{RT}(g,\vartheta)$ mod $\mathcal{P}(f)$.

Definition 6.5 *Let $A \subset \vec{\epsilon}_{x,\lambda}(\vartheta)$ be a finite dimensional vector space and A^+ a direct complement in $\vec{\epsilon}_{x,\lambda}(\vartheta)$. A germ $f \in \vec{\epsilon}_{x,\lambda}(\vartheta)$ is called* A-A^+-determined, *if $f = f_A + f_{A^+}, f_A \in A, f_{A^+} \in A^+$ and there exists a $(S,X,\Lambda) \in \mathcal{B}$ with $(S,X,\Lambda)f = f_A$.*

Lemma 6.6 *Let $f \in \vec{\epsilon}_{x,\lambda}(\vartheta)$ and V a direct complement of $\mathcal{P}(f)$ in $\vec{\epsilon}_{x,\lambda}(\vartheta)$. Then all $g \in \mathcal{O}_f$ are V-$\mathcal{P}(f)$-determined.*

Proof: The statement follows directly from the definition of $\mathcal{P}(f)$. $\square$

Lemma 6.6 means that the action of $\mathcal{B}$ on f can be restricted to the action on $V \cap \mathcal{O}_f$. Definition 6.5 is especially important for certain cases. In order to investigate this we need some further definitions.

Let $\iota_1 : \mathbf{R}[x,\lambda]^\vartheta_\vartheta \to \vec{\epsilon}_{x,\lambda}(\vartheta)$ be the embedding of equivariant polynomial mappings into the space of germs. The mapping $\pi_1 : \vec{\epsilon}_{x,\lambda}(\vartheta) \to \mathbf{R}[x,\lambda]^\vartheta_\vartheta$ is the inverse and is defined in the following way: If a C^∞-mapping g as a representative of the germ mapping $[g]$ is polynomial, then $\pi_1([g]) = g$. If no polynomial representative exists, then $\pi_1([g]) = 0$. Let $P_k^\vartheta \subset \mathbf{R}[x,\lambda]^\vartheta_\vartheta$ denote the subset of all equivariants up to degree k. Let $J^k : \vec{\epsilon}_{x,\lambda}(\vartheta) \to P_k^\vartheta$ denote the Taylor expansion up to degree k and let $j^k : \vec{\epsilon}_{x,\lambda}(\vartheta) \to \vec{\epsilon}_{x,\lambda}(\vartheta)$ be the mapping $[\cdot] \circ J^k$ with image $[P_k^\vartheta]$. Let $\vec{\mathcal{M}}_k$ denote the submodule of $\vec{\epsilon}_{x,\lambda}(\vartheta)$ containing all g with $J^{k-1}g = 0$. Observe that $\vec{\mathcal{M}}_k$ is intrinsic.

Substituting $A = \iota_1(P_k^\vartheta)$ and $A^+ = \vec{\mathcal{M}}_{k+1}$ in Definition 6.5 gives the definition of *k-determinacy* in Mather [16].

Definition 6.7 *(Mather [16]) The germ mapping $f \in \vec{\epsilon}_{x,\lambda}(\vartheta)$ is called k*-determined *if there exists a $g \in P_k^\vartheta$ and $(S,X,\Lambda) \in \mathcal{B}$ such that $g = J^k(f)$ and $(S,X,\Lambda)f = \iota_1(g)$. f is called* finitely determined *if it is k-determined for some $k \in \mathbf{N}$.*

For sake of completeness we state:

Lemma 6.8 *If $f \in \vec{\epsilon}_{x,\lambda}(\vartheta)$ has finite codimension, then there exists a $k \in \mathbf{N}$ such that $\vec{\mathcal{M}}_k \subseteq \mathcal{P}(f)$.*

Proof: Lemma 6.4 gives that $\mathcal{P}(f)$ has finite codimension. $\square$

Lemma 6.9 *$f \in \vec{\epsilon}_{x,\lambda}(\vartheta)$ is of finite codimension if and only if f is finitely determined. Moreover if $f \in \vec{\epsilon}_{x,\lambda}(\vartheta)$ is finitely determined then all $g \in \mathcal{O}_f$ and also the common intrinsic part $\mathcal{P}(f)$ have finite codimension.*

Proof: If f has finite codimension then $\mathcal{P}(f)$ has finite codimension and there exists a $k \in \mathbf{N}$ with $\vec{\mathcal{M}}_{k+1} \subset \mathcal{P}(f)$. Since f is V-$\mathcal{P}(f)$-determined, it is P_k^ϑ-$\vec{\mathcal{M}}_{k+1}$-determined which equals k-determinacy. If f is finitely determined then f has finite codimension which implies that $\mathcal{P}(f)$ has finite codimension. Also for all $g \in \mathcal{O}_f : \mathcal{P}(f) = \mathcal{P}(g) = \dim f < \infty$. $\square$

Corollary 6.10 *Let $f \in \vec{\epsilon}_{x,\lambda}(\vartheta)$ have finite codimension and $\vec{\mathcal{M}}_{k+1} \subset \mathcal{P}(f)$. Then there exists a complement $V \subset \iota_1(P_k^\vartheta)$ of $\mathcal{P}(f)$ in $\vec{\epsilon}_{x,\lambda}(\vartheta)$.*

Corollary 6.10 implies that the investigation of $\mathcal{RT}(f,\vartheta)$ for f of finite codimension can be restricted to the computation in P_k^ϑ. It suffices to use polynomial mappings up to degree k.

7 Germs versus Polynomials

It is clear that the computer cannot handle germs. The closest mathematical object to work with are polynomials, but nevertheless these are completely different objects.

First the polynomials $g \in K[x]$ are identified with C^∞-mappings $\tilde{g} : \mathbf{R}^n \to \mathbf{R}^n \quad x \longmapsto \tilde{g}(x)$ where we assume $K \subset \mathbf{R}$. This embedding is very natural, but the main difference is that $\tilde{g}$ has domain and range which are normally not considered by Computer Algebra

packages. In Computer Algebra polynomials g are handled with the algebraic properties as objects in $K[x]$. The C^∞-mappings $\tilde{g}$ are not considered nor the germs $[\tilde{g}]$.
For the representation and substitution we write

$$\begin{array}{llcl} \pi_2 : \mathbf{R}[x,\lambda]^\vartheta_\vartheta \to H_1(\mathbf{R}[(\sigma,\lambda),b]), & f(x,\lambda) & \to & \sum_{j=1}^r A_j(\sigma,\lambda)b_j, \\ \iota_2 : H_1(\mathbf{R}[(\sigma,\lambda),b]) \to \mathbf{R}[x,\lambda]^\vartheta_\vartheta, & \sum_{j=1}^r A_j(\sigma,\lambda)b_j & \to & f(x,\lambda) = \sum_{j=1}^r A_j(\sigma(x),\lambda)b_j(x). \end{array}$$

Since we compute not with $\mathbf{R}$, but with $K \subset \mathbf{R}$, we consider the embedding

$$\iota_3 \;:\; H_1(K[(\sigma,\lambda),b]) \to H_1(\mathbf{R}[(\sigma,\lambda),b]) \quad \text{and} \tag{19}$$

$$\pi_3 \;:\; H_1(\mathbf{R}[(\sigma,\lambda),b]) \to H_1(K[(\sigma,\lambda),b]), \tag{20}$$

$\pi_3(p) = 0$ if one coefficient in $\mathbf{R} \setminus K$ appears.

Altogether we consider $\iota = \iota_1 \circ \iota_2 \circ \iota_3$ and $\pi = \pi_3 \circ \pi_2 \circ \pi_1$. Unfortunately π is not a module homomorphism, but this property is not needed. For modules $B \subset \vec{\epsilon}_{x,\lambda}(\vartheta)$ the spaces $\pi(B)$ are submodules again, but now over the ring $K[\sigma,\lambda]$.

Another mapping

$$\zeta^k : \vec{\epsilon}_{x,\lambda}(\vartheta) \to H_1(K[(\sigma,\lambda),b]) \tag{21}$$

is given by a Taylor expansion up to order k (the so-called *k-jet*) and then the equivariant in $\mathbf{R}[x,\lambda]^\vartheta_\vartheta$ is expressed in variables b_i, σ_i, λ, i.e. $\zeta^k = \pi_3 \circ \pi_2 \circ J^k$.

Having in mind that Computer Algebra deals with polynomial rings we justify our algorithm with the results of section 6. We have seen that every bifurcation problem f of finite codimension is finitely determined. There exists a $k \in \mathbf{N}$ and a $g \in \iota_1(P_k^\vartheta) \cap \mathcal{O}_f$ such that $\vec{\mathcal{M}}_{k+1} \subset \mathcal{P}(f)$ and $J^l(g) - J^k(f) = 0 \,\forall\, l > k$. So it makes sense to work with $A := \pi_3 \circ \pi_2(P_k^\vartheta)$.

Assume we make an Ansatz $A := \pi_3 \circ \pi_2(P_k^\vartheta)$ for some given k we have the chance to find representatives $g \in \mathcal{O}_f$ in $\pi_2(P_k^\vartheta)$ for all bifurcation problems f of finite codimension which are k-determined.

The second point is the use of $\mathcal{RT}(f,\vartheta)$ mod $\mathcal{P}(f)$. Unfortunately, it is not possible to do a classification of possible intrinsic modules $\mathcal{P}(f)$. On the other hand, for f of finite codimension there exists a k such that $\vec{\mathcal{M}}_{k+1} \subset \mathcal{P}(f)$ and $\mathcal{RT}(f,\vartheta) = V_f + \mathcal{P}(f)$ can be chosen in a way that $V_f \subset \iota_1(P_k^\vartheta)$. Thus it makes sense to replace $\mathcal{RT}(f,\vartheta)$ by $\pi_2(V_f) \subset \pi_2(P_k^\vartheta)$ and/or by the $\mathbf{R}[\sigma,\lambda]$-module generated by $\pi_2(V_f)$. This submodule equals $\pi_2 \circ \pi_3(\mathcal{RT}(f,\vartheta))$. That is why we consider submodules rt in $H_1(K[(\sigma,\lambda),b])$ up to a certain codimension.

Certainly, there are $K[\sigma,\lambda]$-modules rt and possible tangent spaces $t \subset H_1(K[(\sigma,\lambda),b])$ such that no germ $g \in \vec{\epsilon}_{x,\lambda}(\vartheta)$ exists with $\pi(\mathcal{RT}(g,\vartheta)) = rt$ and $\pi(\mathcal{T}(g,\vartheta)) = t$. This follows from the fact that $\mathcal{RT}(g,\vartheta)$ is a $\epsilon_{x,\lambda}(\vartheta)$-module, but rt is a module over the polynomial ring $K[\sigma,\lambda]$. For every $f \in \vec{\epsilon}_{x,\lambda}(\vartheta)$ of finite codimension it holds that $\pi_2 \circ \pi_3(\mathcal{RT}(f,\vartheta))$ has finite codimension, since $\mathcal{RT}(f,\vartheta) = \mathcal{P}(f) + V_f$ with $V_f \subset \iota_1(P_k^\vartheta)$ for some $k \in \mathbf{N}$ as described in section 6.

When we are dealing with tangent spaces we should be aware that there are at least four versions of the restricted tangent space.
For $g \in A \subset H_1(K[(\sigma,\lambda),b])$ there is $RT(g,\vartheta) \subset H_1(K[(\sigma,\lambda),b])$, as defined in Definition 5.1, the space of germ mappings $\mathcal{RT}(\iota(g),\vartheta) \subset \vec{\epsilon}_{x,\lambda}(\vartheta)$, $\pi(\mathcal{RT}(\iota(g),\vartheta) \subset H_1(K[(\sigma,\lambda),b])$ and $K[\sigma,\lambda] \cdot \zeta^k(\mathcal{RT}(\iota(g),\vartheta)) \subset H_1(K[(\sigma,\lambda),b])$. In general the first, the third and the fourth are different.

So concerning our concept there is one open question to which the consideration above give only partial answers.

Is there for every $g \in \vec{\epsilon}_{x,\lambda}(\vartheta)$ a $p \in A$ such that $RT(p,\vartheta) \subseteq \pi(\mathcal{RT}(g,\vartheta)), T(p,\vartheta) \subset \pi(\mathcal{T}(g,\vartheta))$ and $\iota(p) \in \mathcal{O}_g$?

In general $p \in A$ is not unique.

Some known facts about our algorithm can be formulated as follows.

1.) Each $g \in A$ corresponds to $\iota(g)$ which is a member of a contact equivalence class $\mathcal{O}_{\iota(g)}$.

2.) If $RT(g,\vartheta)$ for $g \in A$ has finite codimension, then $\iota(g)$ has finite codimension.

3.) For $g \in A$ the relations $RT(g,\vartheta) \subset \pi(\mathcal{RT}(\iota(g),\vartheta))$, $T(g,\vartheta) \subset \pi(\mathcal{T}(\iota(g),\vartheta))$ hold.

4.) The Ansatzes should be in a way that $P_k \subset A$ and $\vec{M}_k \subset rt$ fit together.

5.) Non-uniqueness of the approach.
Given a Gröbner basis of the $K[\sigma,\lambda]$-module rt and let q complete the possible tangent space $t = rt + K[\lambda] \cdot \{q\}$. The Ansatz (12) $g \in A$ leads with the conditions (15) or (16) to a system of equations in the parameters. So the question whether there exist a $g \in A$ satisfying the conditions is related to the question whether the system of equations has a solution in K. It is clear that there may be several solutions of the system of equations. This reflects the fact that we are testing for inclusion, but not equality. The second reason for non-uniqueness is that there may be several representatives $g \in A$ with $\iota(g) \in \mathcal{O}$ of the same orbit satisfying (15) (or (16)) e.g. the equations depend not on all the parameters.

6.) While for all $g \in \mathcal{O}_{\iota(f)}$ for some given $f \in A$ the restricted tangent spaces $\mathcal{RT}(g,\vartheta)$ have the same codimension, this is not necessarily true for all $\pi(\mathcal{RT}(g,\vartheta))$ where $g \in \pi(\mathcal{O}_{\iota(g)}) \cap A$.

The difference between germs and polynomials influences all computations, i.e. the computation of a Gröbner basis. Let $p_1,\ldots,p_m \in K[x,\lambda]^{\vartheta}_{\vartheta}$ generate a module B_k over $K[x,\lambda]_{\vartheta}$. Then $\iota(p_1),\ldots,\iota(p_m)$ generate a module B_ϵ over $\epsilon_{x,\lambda}(\vartheta)$ as well. If $q(x,\lambda) \in K[x,\lambda]_{\vartheta}$ with $q(0,0) \neq 0$ and $q \cdot r \in B_k$, then $[\iota(q\cdot r)\cdot \frac{1}{\iota(q)}] = \iota(r) \in B_\epsilon$, but $r \in B_k$ is not true in general. This argumentation was taken from [1].

For the computation of the codimension of the module $\pi(\epsilon_{x,\lambda}(\vartheta)\cdot\iota(B))$, $B \subset H_1(K[(\sigma,\lambda),b])$ one proceeds as follows: Given a set of generators of B one computes the Gröbner basis. If a polynomial of type $b_i p(\sigma,\lambda)q(\sigma,\lambda)$ with $p(0,0) \neq 0$ is an element of the Gröbner basis then recompute the Gröbner basis with input old Gröbner basis plus $b_i q(\sigma,\lambda)$. Eventually this step has to be repeated. Finally, one gets an approximation of $\pi(\epsilon_{x,\lambda}(\vartheta)\cdot\iota(B))$ and a bound for its codimension. In order to get $\pi(\epsilon_{x,\lambda}(\vartheta)\cdot\iota(B))$ precisely one needs to check *all* polynomials in the submodule for factorization which is not possible.

When talking about germs the concept of determinacy is extremely important. The reason is the following: If one has computed the k-jet of a k-determined germ, no higher order terms in the Taylor expansion are necessary. The essential phenomena are captured.

Our approach is completely different and does not determine the order of determinacy since Computer Algebra methods are not able to determine the intrinsic module $\mathcal{P}(f)$ nor the order k with $\vec{\mathcal{M}}_k(\vartheta) \subset \mathcal{P}(f)$. Of course for a given $g \in A$ there exists in general not a direct complement A^+ of $\mathbf{R}\cdot\iota(A)$ in $\vec{\epsilon}_{x,\lambda}(\vartheta)$ such that g is A-A^+-determined. But the $g \in A$ which we compute fulfill special requirements.

Concerning determinacy the following questions arise. Let $g \in A$ be given.

(a) Are there other representatives of g in A? What is $\pi(\mathcal{O}_{\iota(g)}) \cap A$?

(b) Assume $\iota(g)$ has finite codimension and g has degree k. Is g k-determined?

The last point in this section is the use or non-use of standard bases as investigated by Mora [18]. His tangent cone algorithm is implemented in Singular [13]. These bases correspond in principle much better to singularities than Gröbner bases, since the lowest terms play the role of the leading terms. On the other hand an Ansatz as in (13) is not possible since the number of terms in a standard bases are not known. Secondly, as outlined in Section 6 it suffices to limit the degree to k for a k-determined germ. These are the reasons why we use Gröbner bases instead of standard bases.

8 Generation of Possible Tangent Spaces

The main idea of this paper is to classify normal forms by a classification of possible tangent spaces. The handling of possible tangent spaces $t = rt + K[\lambda]\{q\} \subset H_1(K[(\sigma,\lambda), b])$ is possible by using a Gröbner basis of the $K[\sigma,\lambda]$-module rt and the additional element q.

Algorithm 8.1 generates leading terms of Gröbner bases of $K[\sigma,\lambda]$-modules with codimension less than a given number. Since $\mathcal{RT}(g,\vartheta)$ has finite codimension if and only if $\mathcal{T}(g,\vartheta)$ is of finite codimension it is reasonable to consider such rt that have finite codimension. For practical reasons the list of these modules (represented by the leading terms of the Gröbner bases) is not complete by restricting the recursion deepness in Algorithm 8.1. The function *successor* generates the following monomial with the next lowest ordering.

Algorithm 8.1 *(Generate LT's of Gröbner bases of $K[\sigma,\lambda]$-modules in $H_1(K[(\sigma,\lambda), b])$)*
Input: *module variables b_i, variables σ, λ,*
maximal codimension codim*, vector $d \in \mathbf{N}^n$ for weighted ordering*

GenerateLts(nil,m_0, codim, 1, 0) % *m_0 is the monomial with the lowest ordering.*

procedure GenerateLts(lts, m, codim, tiefe, h)
if Codimension(lts)> codim then return nil;
if FiniteCodimensionp(lts) then return list(lts);
if tiefe> maxtiefe then return nil;
if m element of $K[\sigma,\lambda]$-module generated by lts
then return GenerateLts(lts,successor(m), codim, tiefe + 1, h)
else
return append
(GenerateLts(lts $\cup \{m\}$,successor(m), codim, tiefe + 1, h),
GenerateLts(lts,successor(m), codim, tiefe + 1, $h + 1$));

The basic idea of Algorithm 8.1 is to run through the monomials starting with the first one and decide whether it belongs to the leading terms or not.

Then the classification of possible tangent spaces is completed by choosing a leading term for q. This is either zero or a monomial which is not an element of $K[\sigma,\lambda] \cdot LT(rt)$. In principle one could eliminate some of the approaches if they generate the same $K[\lambda]$-module. But the introduction of the parameters and the effect of Theorem 8.2 is different.

Once the leading terms $LT = \{m_1, \ldots, m_\nu\}$ in $H_1(K[(\sigma,\lambda), b])$ are given such that they form a Gröbner basis and r in $H_1(K[(\sigma,\lambda), b])$ is given such that either $r = 0$ or $r \notin K[\sigma,\lambda] \cdot LT)$, an Ansatz is made:

$$P_i = m_i + \sum_{m <_T m_i, m \notin K[\sigma,\lambda]\cdot LT} p(m_i, m) \cdot m, \quad i = 1, \ldots, \nu, \tag{22}$$

$$Q = r + \sum_{m <_T r, m \notin K[\sigma,\lambda]\cdot LT} p(r, m) \cdot m \tag{23}$$

The restrictions $m \notin K[\sigma,\lambda] \cdot LT$ can be assumed since otherwise by linear combination one can always achieve $p(m_i, m) = 0$. In other words we assume a reduced Gröbner basis. Further restriction of the unknowns are given by the fact that a space $\mathcal{RT}(\iota(g), \vartheta)$ has to be approximated. So one has to take the relations between germs and polynomials into account.

The following theorem relies heavily on the concept of an intrinsic module $\mathcal{P}(f)$.

Theorem 8.2 *Let $g \in \vec{\epsilon}_{x,\lambda}(\vartheta)$ have finite codimension. Let $RT \subset \pi(\mathcal{RT}(g,\vartheta))$, $T \subset \pi(\mathcal{T}(g,\vartheta))$ be modules over $K[\sigma,\lambda]$ and $K[\lambda]$, respectively. Let $F = \{P_1 \ldots, P_\nu\}$ and $LT = LT(F)$ with*

$$P_i = m_i + \sum_{m <_T m_i, m \notin K[\sigma,\lambda]\cdot LT} p(m_i, m) m, \quad p(m_i, m) \in K, \quad i = 1, \ldots, \nu.$$

Let F be a Gröbner basis of RT and

$$Q = r + \sum_{m <_T r, m \notin K[\sigma,\lambda]\cdot LT} p(m_i, m) m,$$

such that $T = RT + K[\lambda] \cdot \{Q\}$.

(i) Consider the polynomial P_i and a monomial $\tilde{m} \notin K[\sigma,\lambda] \cdot LT$ such that there exists a monomial $m_0 \notin K$ with $m_i = m_0 \cdot \tilde{m}$. If $p(m_i, \tilde{m}) \neq 0$ then

$$\frac{1}{\iota(m_0 + p(m_i, \tilde{m}))} \iota(P_i) \in \mathcal{RT}(g, \lambda),$$

and there is an element in $\pi(\mathcal{RT}(g,\vartheta))$ of the form

$$\sum_{m \notin K[\sigma,\lambda]\cdot LT} \alpha_m \cdot m, \quad \alpha_m \in K. \tag{24}$$

(ii) Consider the polynomials P_i, P_j and a $\tilde{m} \in K[\sigma,\lambda]$ with $\tilde{m} < m_j < m_i$ and $m_j = m_0 \cdot \tilde{m}, m_0 \neq 1$. If $p(m_i, \tilde{m}) \neq 0$ then

$$\frac{1}{\iota(m_0 + p(m_i, \tilde{m}))} \iota(P_i + P_j) \in \mathcal{RT}(g, \lambda),$$

and $\pi(\mathcal{RT}(g,\vartheta))$ contains an element of the form (24).

(iii) Let $\tilde{m} < m_i$ with $r = m_0 \cdot \tilde{m}$ with $m_0 = \lambda^j, j \neq 0$ Then $p(m_i, \tilde{m}) = 0$ or

$$\frac{1}{\iota(m_0 + p(m_i, \tilde{m}))} \iota(P_i + Q)) \in \mathcal{T}(g, \vartheta),$$

and $\pi(\mathcal{T}(g,\vartheta))$ contains a polynomial of the form

$$\sum_{m \notin K[\sigma,\lambda]\cdot LT, m \notin K[\lambda] r} \alpha_m \cdot m, \quad \alpha_m \in K. \tag{25}$$

(iv) *Let $\tilde{m} < r$ with $m_i = m_0 \cdot \tilde{m}, m_0 = \lambda^j, j \geq 1$, $m_i < r$. Then $p(r, \tilde{m}) = 0$ or*

$$\frac{1}{\iota(m_0 + p(r, \tilde{m}))}\iota(Q + P_i)) \in \pi(\mathcal{T}(g, \vartheta)),$$

$\pi(\mathcal{T}(g, \vartheta))$ contains a polynomial of the form (25).

(v) *Let $\hat{m} <_T \tilde{m} <_T m_i$ with $\tilde{m} = m_0 \cdot \hat{m}, m_0 \neq 1$. Then either $p(m_i, \hat{m}) = 0$ or*

$$\frac{1}{\iota(p(m_i, \tilde{m})m_0 + p(m_i, \hat{m}))}\iota(P_i) \in \pi(\mathcal{RT}(g, \vartheta)),$$

and $\pi(\mathcal{RT}(g, \vartheta))$ contains a polynomial of the form (24).

(vi) *Let $\hat{m} <_T \tilde{m} \leq_T r$ with $\tilde{m} = m_0 \cdot \hat{m}, m_0 \in K[\lambda], m_0 \neq 1$. Then either $p(r, \hat{m}) = 0$ or*

$$\frac{1}{\iota(p(r, \tilde{m})m_0 + p(r, \hat{m}))}\iota(Q) \in \pi(\mathcal{T}(g, \vartheta)),$$

and $\pi(\mathcal{T}(g, \vartheta))$ contains a polynomial of the form (25).

Proof: Since the proofs are all very similar we prove only (i). RT is a module over $K[\sigma, \lambda]$. The embedding $\iota(RT)$ generates a module over $\epsilon_{x,\lambda}(\vartheta)$ and is a submodule of $\mathcal{RT}(g, \vartheta)$. Since $p(m_i, \tilde{m}) \neq 0$, we can embed $\frac{1}{m_0 + p(m_i, \tilde{m})}$ as a germ in $\epsilon_{x,\lambda}(\vartheta)$ and thus $\iota(\frac{P_i}{m_0 + p(m_i, \tilde{m})})$ is an element of the submodule generated by $\iota(RT)$. This proves the first statement. The explicit form is

$$h := \frac{\iota(P_i)}{\iota(m_0 + p(m_i, \tilde{m}))} = \iota(\tilde{m}) + \iota\left(\sum_{m <_T m_i, m \neq \tilde{m}, m \notin K[\sigma, \lambda] \cdot LT} \frac{p(m_i, m) \cdot m}{m_0 + p(m_i, \tilde{m})}\right).$$

$\mathcal{RT}(g, \vartheta)$ can be written as $\mathcal{RT}(g, \vartheta) = \mathcal{P}(g) + V_g$ where $\mathcal{P}(g)$ is the intrinsic module and where we can assume $V_g \subset \iota_1(\mathcal{P}_k^\vartheta)$ for some $k \in \mathbf{N}$. Since $h \in \mathcal{RT}(g, \vartheta)$ there exists a $\tilde{h} \in \mathcal{P}(g)$ such that $h + \tilde{h} \in V_g \subset \iota_1(\mathcal{P}_k^\vartheta)$. Then $\pi(h + \tilde{h}) \in H_1(K[(\sigma, \lambda), b])$ is of weighted degree $\leq k$. Reduction modulo the Gröbner basis gives a polynomial of the desired form. □

By Theorem 8.2 we can restrict the number of parameters in Ansatz (13) a lot. Since the Ansatz is assumed to approximate $\mathcal{RT}(g, \vartheta)$ and $\mathcal{T}(g, \vartheta)$ and it is unlikely to happen that the polynomials (24) and (25) are zero we can assume $p(m_i, m) = 0, p(r, m) = 0$ for all cases discussed in Theorem 8.2. We did not investigate the circumstances under which (24) and (25) are zero or not. The cases $p(m_i, m) \neq 0$ leads to an Ansatz with a different set of leading terms of the Gröbner basis.

Although we have thus chosen a lot of the free parameters in the Ansatz to be zero one has to make sure that the Ansatz (13) defines a Gröbner basis for all values of the remaining parameters. Due to Theorem 3.2 one needs to demand that S-polynomials reduce to zero

$$S(P_i, P_j) \to_F^* 0 \quad \text{in} \quad H_1(K[p][(\sigma, \lambda), b]) \tag{26}$$
$$S(P_i, Q) \to_{F,Q}^+ 0 \quad \text{in the} \quad K[p][\lambda] - \text{module generated by } H_1(K[p][\lambda, u]), \tag{27}$$

where u stands for all monomials in $H_1(K[\sigma, b])$ and $F = \{P_1, \ldots, P_\nu\}$. Due to the module version of the Buchberger algorithm (see section 3) one needs to consider all

$S(P_i, P_j)$ where a k exists with $b_k|lt(P_i)$ and $b_k|lt(P_j)$. In (27) all P_i with $lt(Q)|lt(P_i)$ are considered. Top reductions $\rightarrow_F$ are carried out for $S(P_i, P_j)$ and $S(P_i, Q)$ successively until a normal form is obtained (command Preduce in REDUCE). The normal forms of $S(P_i, Q)$ are then further reduced with top reductions $\rightarrow_Q$ until a normal form in $H_1(K[p][\lambda, u])$ is obtained. Demanding the coefficients of the normal forms to be zero gives the equations for the unknowns $p(m_i, m), p(r, m)$. The equations turned out to be very small in the computed examples and could be solved by the REDUCE command Solve. The result is substituted into (13) and (14) completing the Ansatz.

Before comparing the possible tangent spaces with the tangent space $T(g, \vartheta)$ for $g \in A$ given by the Ansatz (12), the computation of $T(g, \vartheta)$ has to be discussed in detail since this involves calculations with the old variables $x_1, \ldots, x_n$. Recall that for $g(b, \sigma, \lambda) = \sum_{i=1}^r b_i A_i(\sigma, \lambda)$ the restricted tangent space is generated by

$$S_i(x)g, i = 1, \ldots, l, d_x g \cdot b_j(x), j = 1, \ldots, r, \tag{28}$$

and completed by g_λ to $T(g, \vartheta)$.

The treatment of g and $d_x g$ in (28) suffices to substitute $b_i = b_i(x)$ and $g = \sum_{i=1}^r A_i b_i(x)$. Using $\frac{d}{dx}A_i = \frac{d}{d\sigma}A_i \frac{d\sigma}{dx}$ the derivatives $A_{i\sigma_j}$ are treated as kernels in the Computer Algebra package, while σ_{ix_j} is substituted in (28) by its corresponding polynomial. Thus $d_x g \cdot b_j(x)$ yields expressions in $K[x, \lambda]_\vartheta^\vartheta$ as a module over the polynomial ring in A_i and $A_{i\sigma_j}$. Of course A_i and $A_{i\sigma_j}$ appear only linearly.

By an algorithm presented in [10] and mentioned in section 4 these expressions are represented as polynomials in $\oplus_{i=1}^r b_i K[\sigma, \lambda, A_i, A_{i\sigma_j}]$. Also g_λ gives one polynomial in the same space.

The Ansatz (12) is equivalently expressed as an Ansatz $A_i(\sigma, \lambda)$ in explicit polynomials in $H_1(K[(\sigma, \lambda), c])$. Substitution of A_i and $A_{i\sigma_j}, A_{i\lambda}$ yields expressions in $H_1(K[c][(\sigma, \lambda), b])$ generating $RT(g, \vartheta)$ and $T(g, \vartheta)$, respectively. Let the generators of $RT(g, \vartheta)$ denote by $p_i, i = 1, \ldots, s + l$, the additional element of $T(g, \vartheta)$ by q.

The main step of the automatic classification of normal forms is now prepared. Demanding

$$RT(g, \vartheta) \subseteq rt \text{ and } Q \in t \tag{29}$$

yields a system of equations in the parameters $p(m_i, m), p(r, m)$ and c_m in the following way. Let $F = \{P_1, \ldots, P_\nu\}$ denote the Gröbner bases of Ansatz (13), of rt and Q in Ansatz (14) be the additional element for t. Then (29) is equivalent to.

$$p_i \rightarrow_F^* 0 \quad \text{in} \quad H_1(K[p, c][(\sigma, \lambda), b]), \tag{30}$$

$$q \rightarrow_{F,Q}^+ 0 \quad \text{in} \quad H_1(K[p, c][(\sigma, \lambda), b] \text{ and } H_1(K[p, c][\lambda, u], \tag{31}$$

where u stands for all monomials in $H_1(K[\sigma, b])$. This gives equations for the unknown parameters $p(m_i, m), p(r, m)$ and $c_j(m)$, which are linear in $c_j(m)$. A weaker condition is

$$T(g, \vartheta) \subseteq t. \tag{32}$$

This is equivalent to

$$p_i \rightarrow_{F,Q}^+ 0 \text{ and } q \rightarrow_{F,Q}^+ 0 \tag{33}$$

in $H_1(K[p, c][(\sigma, \lambda), b]$ and $H_1(K[p, c][\lambda, u]$. The yielded equations are linear in $c_j(m)$ as well.

The operator Solve in REDUCE solves such systems of equations distinguishing nicely the different cases of parameter values of $p(m_i, m), p(r, m_i)$. This solutions are substituted into g.

In the end of this section we observe the following

a) For each possible tangent space $t = rt + K[\lambda]\{r\}$ there is one or several g.

b) The subset relation in (29) (or (32)) may happen to be a proper inclusion and no equality, since only one direction is used.

c) The coefficients in the normal forms g still depend on part of the unknown parameters $p(m_i, m), p(r, m), c_m$.

Statement b) means that an approach t of codimension d may lead to normal forms of codimension $> d$. Statement c) can be interpreted in a way that the representative of the ϑ-contact equivalence class is not unique. The restriction to one representative is the topic of the next section.

9 Restriction to Special Representatives

Once a $g \in H_1(K[(\sigma, \lambda), b])$ depending on some limited number of parameter is known the parameters are restricted to certain values by the action of the subgroups $\mathcal{B}_{\mathcal{L}^0}, \mathcal{B}_\lambda, \mathcal{B}_x$. Such criteria are developed in this section.

After performing the process described in section 8 we end up with a $g \in A \subset H_1(K[(\sigma, \lambda), b])$ depending on some parameter c_m and $p(m_i, m), p(r, m)$.

Proposition 9.1 *Let $g = \sum_{m \in A} a_m(p, c) \cdot m$ be A-A^+-determined and let the coefficient $a_{\hat{m}}$ be fixed, where $\hat{m} = b_j \cdot \sigma^i \cdot \lambda^k, k \geq 1$. Then $(Id, Id, \Lambda(\lambda)) \in \mathcal{B}, \Lambda(\lambda) = \lambda + 0 \cdot \lambda^2 + \cdots + 0 \cdot \lambda^k + \tilde{\Lambda}(\lambda)$, gives a ϑ-contact equivalence germ in $\mathcal{O}_g(\mathcal{B}_\Lambda)$ and the coefficients $\tilde{a}_m$ of $\pi((Id, Id, \Lambda(\lambda))g) = \sum \tilde{a}_m m \neq 0$ satisfy*

$$\tilde{a}_{\hat{m}} = a_{\hat{m}}, \quad \tilde{a}_m = 0, \text{ for } m = \lambda^\nu \cdot \hat{m}, \nu = 1, 2, \ldots$$

Proof: Substitution of $\Lambda(\lambda)$ into g produces new terms of the form $b_j \sigma^i \lambda^\nu, \nu > k$. Choosing $\tilde{\Lambda}(\lambda)$ in a special way one can always achieve that $\tilde{a}_m = 0$ for $m = b_j \sigma^i \lambda^\nu, \nu = k + 1, \ldots, \mu$. Since g is finitely determined, so is $(Id, Id, \Lambda(\lambda))g$ and another representative in A with the required properties can be constructed. $\Box$

Criterion 9.2 *crithottvar (eliminate higher order terms in λ)*
Input: $g = \sum_{m \in A} a_m(p, c)m$,
monomial $\hat{m} = b_j \sigma^i \lambda^k, k \geq 1$ with fixed coefficient $a_{\hat{m}}$
Output: $g := g - \sum_{m \in A \setminus \{\hat{m}\}, m \in K[\lambda] \cdot \hat{m}} a_m m$

We would like to have more criteria which deletes higher order terms, i.e. show that $g + m \sim g$, where m is a monomial of higher degree.

Definition 9.3 *([12, p. 204 f]) Let $g \in \vec{\epsilon}_{x,\lambda}(\vartheta), b_i(x), i = 1, \ldots, r$ the generators of $\mathbf{R}[x]^\vartheta_\vartheta$ and $S_j, j = 1, \ldots, k$ the generators of the commuting matrices. The submodule of $\vec{\epsilon}_{x,\lambda}(\vartheta)$ generated by $M \cdot RT(g, \vartheta)$, $\frac{d}{dx} g b_i$ with $deg(b_i) \geq 2$ and $S_j g$ with $deg(S_j) \geq 1$ is denoted by $\mathcal{K}_s(g, \vartheta)$. Furthermore one defines $\mathcal{K}(g, \vartheta) = \mathcal{K}_s(g, \vartheta) + \epsilon_\lambda\{\lambda^2 g_\lambda\}$.*

Theorem 9.4 *([12, p. 205 f], Gaffney) Let $p \in$Itr$(\mathcal{K}(g, \vartheta))$. Then $g + p$ is ϑ-contact equivalent to g.*

Since we are not able to determine $\mathrm{Itr}(\mathcal{K}(g, \vartheta))$ with Computer Algebra methods we use a heuristic which is similar to crithottvar (Criterion 9.2).

Criterion 9.5 *H1 (eliminate higher order terms)*
Input: $g = \sum_{m \in A} a_m(p,c)m$,
monomial $\hat{m}$ *with fixed coefficient* $a_{\hat{m}}$
Output: $g := g - \sum_{m \in A \setminus \{\hat{m}\}, m \in K[\sigma,\lambda]\cdot \hat{m}} a_m m$

The following criteria are based on the action of the groups $\mathcal{B}_{\mathcal{L}^0}, \mathcal{B}_\lambda, \mathcal{B}_x$ on g.

Proposition 9.6 *Let* $g = \sum_{m \in A} a_m(p,c)m$, *where the coefficients* a_m *depend on some parameter* p, c. *Let* $S_1, \ldots, S_k \in \mathcal{L}(\vartheta)^0$ *form a basis of the vector space of constant commuting matrices. Assume that for* i *exists some integers* $j(i)$ *such that*

$$S_i b_{j(i)} \in < b_{j(i)} > . \tag{34}$$

Then $(\sum_{i=1}^k \gamma_i S_i)g \in \mathcal{O}_g(\mathcal{B}_{\mathcal{L}^0})$, *where* $\gamma_i > 0, i = 1, \ldots, k$ *and*

$$(\sum_{i=1}^k \gamma_i S_i, Id, Id)g = \sum_i \left(\sum_{m \in A, b_{j(i)} | m} a_m(p,c) \cdot \gamma_i \cdot m \right) + \sum_{m \in A, b_{j(i)} \not| m \forall j(i)} a_m(p,c)m.$$

Condition (34) is a very strict condition, if ϑ decomposes into several irreducible representations. On the other hand it can be easily achieved by choosing the right coordinate system. In this coordinate system the matrices $\vartheta(t)$ have block diagonal form with blocks corresponding to irreducible representations. Then the fundamental equivariants $b_j(x)$ and fundamental commuting matrices $S_i(x)$ can be determined in a way that each depends only on few of the variables x_i explicitly. The algorithmic determination of b_j and S_i fulfills this requirement automatically. So (34) is a natural condition in our implementation.

Criterion 9.7 *critmat (action of* $\mathcal{B}_{\mathcal{L}^0}$ *on* g*)*
Input: $g = \sum_{m \in A} a_m(p,c)m$,
$S_1, \ldots, S_k$, *set of indices* $J_i = \{j(i)\}, i = 1, \ldots, k$.

for $i = 1, \ldots, k$ *determine the monomial* m^i_{min} *with the lowest ordering in*
   $M_i = \{m \in A |\, b_{j(i)}|m$ *for one* $j(i) \in J_i$ *and* a_m *depends on* p *or* $c\}$
   do $g := g - (a_{m^i_{min}} + sign(a_{m^i_{min}}))m^i_{min}$
      apply crithottvar wrt m^i_{min} *to* g
      apply heuristic H1 wrt m^i_{min} *to* g

Proposition 9.8 *Let* A *and* g *as in Proposition 9.6. Let* $\Lambda(\lambda) \equiv \gamma \cdot \lambda, \gamma > 0$. *Then* $(Id, Id, \Lambda)g \in \mathcal{O}_g(\mathcal{B}_\lambda)$ *and*

$$(Id, Id, \Lambda)g = \sum_{m \in A, \lambda \not| m} a_m m + \sum_{i=1}^{maxi} \sum_{m \in A, \lambda^i | m, \lambda^{i+1} \not| m} \gamma^i a_m m.$$

Criterion 9.9 *crittvar (action of* $\mathcal{B}_\lambda$ *on* g*)*
Input: $g = \sum_{m \in A} a_m(p,c)m$,
variable λ
determine the monomial m_{min} *with the lowest ordering in*

$$M = \{m \in A |\, \lambda | m \text{ and } a_m \text{ depends on } p \text{ or } c\}$$

$g := g - (a_{m_{min}} + sign(a_{m_{min}}))m_{min}$
apply crithottvar wrt m^i_{min} *to* g
apply heuristic H1 wrt m_{min} *to* g

Proposition 9.10 *Let A and g as in Proposition 9.6. Let ϑ be block diagonal and let $x = (x_1, \ldots, x_l)$ be grouped in variables corresponding to the irreducible representations in ϑ. Let $b_j(x), j = 1, \ldots, l$ be those homogeneous fundamental equivariants which are linear. Let each b_j correspond to one irreducible representation. Then $(Id, X, Id)g$ with $X = \sum_{j=1}^{l} \gamma_j b_j, \gamma_j > 0$ is an element of $\mathcal{O}_g(\mathcal{B}_x)$ and*

$$(Id, X, Id)g = \sum_{m \in A, \exists b_j : m = b_j \sigma^i \lambda^\nu} \gamma_j \cdot \gamma^{Di} \cdot a_m(p, c) \cdot m + \sum_{m \in A, m = b_k \sigma^i \lambda^\nu, k > l} \gamma^e \cdot \gamma^{Di} \cdot a_m(p, c) \cdot m,$$

where we assume that the fundamental invariants σ_k are homogeneous in each variable group x_j and $D = (d_{kj})$ consists of the degrees d_{kj} of σ_k in the variables x_j. We assume furthermore that the fundamental equivariants $b_k, k > l$ are homogeneous in each variable group x_j and e_j is the degree in x_j.

Criterion 9.11 *critequis (action of $\mathcal{B}_X$ on g)*
Input: $g = \sum_{m \in A} a_m(p, c) m$,
homogeneous, linear equivariants b_j

for $j = 1, \ldots, l$ find the monomial m^j_{min} with the lowest ordering in
$M_j = \{m \in A |\ \lambda | m$ and a_m depends on p or $c\}$
do $g \to g - (a_{m^j_{min}} + sign(a_{m^j_{min}})) m^j_{min}$
apply crithottvar wrt m^j_{min} to g
apply heuristic H1 wrt m^j_{min} to g

Observe that the actions of $\mathcal{B}_{\mathcal{L}^0}, \mathcal{B}_\lambda, \mathcal{B}_x$ are independent of each other and thus can be applied in any order. However, one has to be careful when one applies criteria 9.7, 9.9, and 9.11 one after the other.

Example 9.12 *Let b_j be a linear equivariant and $S_i \in \mathcal{L}(\vartheta)^0$ a constant commuting matrix with $S_i b_j = b_j$. Then $g = a_{b_j} b_j + a_{\lambda b_j} \lambda b_j + r$ is transformed to*

$$(\gamma_i S_i, Id, Id)(Id, \gamma_j b_j, Id)g = \gamma_i \gamma_j (a_{b_j} b_j + a_{\lambda b_j} \lambda b_j) + \tilde{r}.$$

It is not possible to choose both coefficients to be $sign(a_{b_j})$ and $sign(a_{\lambda b_j})$. This consideration becomes important for the group action $Z_2 \times Z_2$, see subsection 10.3.

In the end one obtains a $g \in A$ depending on much less parameter than in the beginning. It is not clear whether for each value of these parameter g is in the same ϑ-contact equivalence class. The parameter (or part of them) may be *moduli* parameter, i.e. g is for different values topological equivalent, but not algebraically equivalent. Some special values may be exceptions for the topological equivalence.

The experience shows that part of this exceptions can be found by computing the Gröbner bases of $RT(g, \vartheta)$ depending on the parameters. The exceptional values for the case that the computed polynomials do not form a Gröbner basis are interesting values, i.e. the topological properties of the solution set of g are exceptional, see e.g. subsection 10.3.

The computation of the Göbner basis gives an approximation of the codimension as well.

An approximation of the *universal unfolding* is given by

$$g + \sum_{m \notin K[\sigma, \lambda] \cdot LT(rt), m \notin K[\lambda] \cdot r} \alpha_m m$$

10 Discussion of Results

In this section several examples for various groups are presented. Since for these group actions several singularities of various codimension are known, this gives the opportunity to compare the results of our algorithm with results from the literature. The first example is discussed in detail in order to show how our algorithm proceeds.

10.1 D_3

We consider $D_3 = \{id, r, r^2, s, sr, sr^2\}$ and its 2-dimensional irreducible representation which is faithful, i.e.

$$\vartheta(r) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad \vartheta(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The fundamental invariants are computed to be

$$\sigma_1(x_1, x_2) = \frac{(x_1^2 + x_2^2)}{2}, \quad \sigma_2(x_1, x_2) = \frac{(-x_1^3 + 3 \cdot x_1 \cdot x_2^2)}{4}.$$

The fundamental equivariants are determined to be

$$b_1(x_1, x_2) = \frac{1}{2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad b_1(x_1, x_2) = \frac{1}{4}\begin{pmatrix} -x_1^2 + x_2^2 \\ 2 \cdot x_1 \cdot x_2 \end{pmatrix}.$$

This corresponds to the results on 2-dimensional irreducible representations of the dihedral groups D_h, [12, p. 45]. The generators of the free module of commuting matrices are calculated to be

$$S_1(x) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_3(x) = \frac{1}{8}\begin{pmatrix} 3x_1^2 + x_2^2 & 2x_1x_2 \\ 2x_1x_2 & x_1^2 + 3x_2^2 \end{pmatrix},$$

$$S_2(x) = \frac{1}{4}\begin{pmatrix} -x_1 & x_2 \\ x_2 & x_1 \end{pmatrix}, \quad S_4(x) = \frac{1}{16}\begin{pmatrix} -3x_1^3 - 3x_1x_2^2 & -3x_1^2x_2 + 5x_2^3 \\ 9x_1^2x_2 + x_2^3 & 3x_1^3 + 3x_1x_2^2 \end{pmatrix},$$

As described in the previous sections we do not compute in the coordinates x_1, x_2, λ, but in $b_1, b_2, \sigma_1, \sigma_2, \lambda$. The term ordering $>_{D_3}$ is defined by the variable ordering $b_2 > b_1 > \sigma_2 > \sigma_1 > \lambda$ and the weights

$$(\deg(b_2(x)), \deg(b_1(x)), \deg(\sigma_2(x)), \deg(\sigma_1(x)), 1) = (2, 1, 3, 2, 1).$$

In order to compute in $H_1(K[(\sigma_2, \sigma_1, \lambda), (b_2, b_1)])$ the grading with $(1, 1, 0, 0, 0)$ is used first and combined with $>_{D_3}$. The condition (34) in Proposition 9.6 is trivially fulfilled because the group action is irreducible. The Ansatz space A of possible bifurcation problems is chosen to be generated by 14 monomials such that

$$g = \sum_m (c_1(m)b_1 + c_2(m)b_2) = A_1(\sigma, \lambda)b_1 + A_2(\sigma, \lambda)b_2$$

where the sum runs over the first 7 monomials (in the weighted ordering) and $c_1(m)$ and $c_2(m)$ are unknowns. Since we investigate bifurcation problems we set $c_1(1) = 0$.

The kernels A_1 and A_2 are still used to compute the tangent space $T(g,\vartheta)$ as defined in Definition 5.1. The restricted tangent space is generated by

$$\begin{aligned} rt \;=\; \{ & \tfrac{2A_1b_2+4(A_1)_{\sigma_1}b_1\sigma_2+3(A_1)_{\sigma_2}b_1\sigma_1^2+2A_2b_1\sigma_1+4(A_2)_{\sigma_1}b_2\sigma_2+3(A_2)_{\sigma_2}b_2\sigma_1^2}{4}, \\ & \tfrac{A_1b_1+2(A_1)_{\sigma_1}b_1\sigma_1+3(A_1)_{\sigma_2}b_1\sigma_2+2A_2b_2+2(A_2)_{\sigma_1}b_2\sigma_1+3(A_2)_{\sigma_2}b_2\sigma_2}{2}, \\ & \tfrac{A_1b_1+A_2b_2}{2}, \quad \tfrac{2A_1b_2+A_2b_1\sigma_1}{4}, \quad \tfrac{3A_1b_1\sigma_1+2A_2b_1\sigma_2+A_2b_2\sigma_1}{4}, \\ & \tfrac{4A_1b_1\sigma_2+2A_1b_2\sigma_1+5A_2b_1\sigma_1^2-4A_2b_2\sigma_2}{8} \} \end{aligned}$$

and $Q = (A_1)_\lambda b_1 + (A_2)_\lambda b_2$ completes the tangent space. Then A_1, A_2 and their derivatives are substituted. Ansatzes $t = rt + K[\lambda]\{q\}$ for the tangent space are generated of codimension $0,1,2,3$ where a recursion deepness of 10 was used. There are two choices how to derive conditions for the parameters $c_1(m), c_2(m)$ and $p(m_i,m)$. Here we use the requirement

$$T(g,\vartheta) \subseteq t.$$

Then the operator SOLVE gives the solutions of the system of equations which are linear in $c_1(m), c_2(m)$. Eventually several cases ($p(m_i,m)=0$, $p(m_i,m)\neq 0$) are distinguished.

Thus simpler (possibly several) polynomials $g \in H_1(K[p][(\sigma_2,\sigma_1,\lambda),(b_2,b_1)])$ are obtained where p stands for $c_1(m), c_2(m)$ and $p(m_i,m)$. The criteria are applied to g in the following order: critequis, critmat, crittvar, deletehots.

Let us discuss the results in detail:

codim≥ 0

$t = rt + k[\lambda]\{q\}$ with $rt = \{b_2, b_1\}, q = 0$

$$g = c_1(\sigma_1)b_1\sigma_1 + c_1(\sigma_2)b_1\sigma_2 + \mathrm{sign}(c_1(\lambda))b_1\lambda + \mathrm{sign}(c_2(1))b_2,$$

g is a typically transcritical bifurcation. For all values of $c_1(\sigma_1)$ and $c_1(\sigma_2)$ the normal form g is in the same contact equivalence class. This shows that the resulting singularities are non-unique.

Also for three other Ansatzes t of codimension 0 the same g is derived.

codim≥ 1

1.) $rt = \{b_1\sigma_2, b_1\sigma_1, b_2, b_1\lambda\}$, $q = 0$
The condition $T(g,\vartheta) \subseteq t$ gives $c_1(\lambda) = 0$. Finally

$$\begin{aligned} g_{111} = \;& c_1(\sigma_1)b_1\sigma_1 + c_1(\sigma_2)b_1\sigma_2 \\ & + \mathrm{sign}(c_1(\lambda^2))b_1\lambda^2 + \mathrm{sign}(c_1(\lambda\sigma_1))b_1\lambda\sigma_1 + \mathrm{sign}(c_2(1))b_2. \end{aligned}$$

This is pictured for the parameter values $c_1(\sigma_1) = 1, c_1(\sigma_2) = 1, \mathrm{sign}(c_1(\lambda^2)) = 1, \mathrm{sign}(c_1(\lambda\sigma_1)) = 1, \mathrm{sign}(c_2(1)) = -1$ in Figure 1. The solutions of $g = 0$ within the fixed point spaces D_3 and Z_2 ($x_2 = 0$) are drawn. The implicit plot routine of Maple [6] was used.

2.) $LT(rt) = \{b_1\sigma_2, b_1\sigma_1, b_1\lambda^2, b_2\}$, $q = b_1\lambda$
Parameters are introduced in the following way:

$$rt = \{b_1\sigma_2, b_1\sigma_1, b_1\lambda^2, b_2 + p(b_2,b_1)\cdot b_1\}, \quad q = b_1\lambda,$$

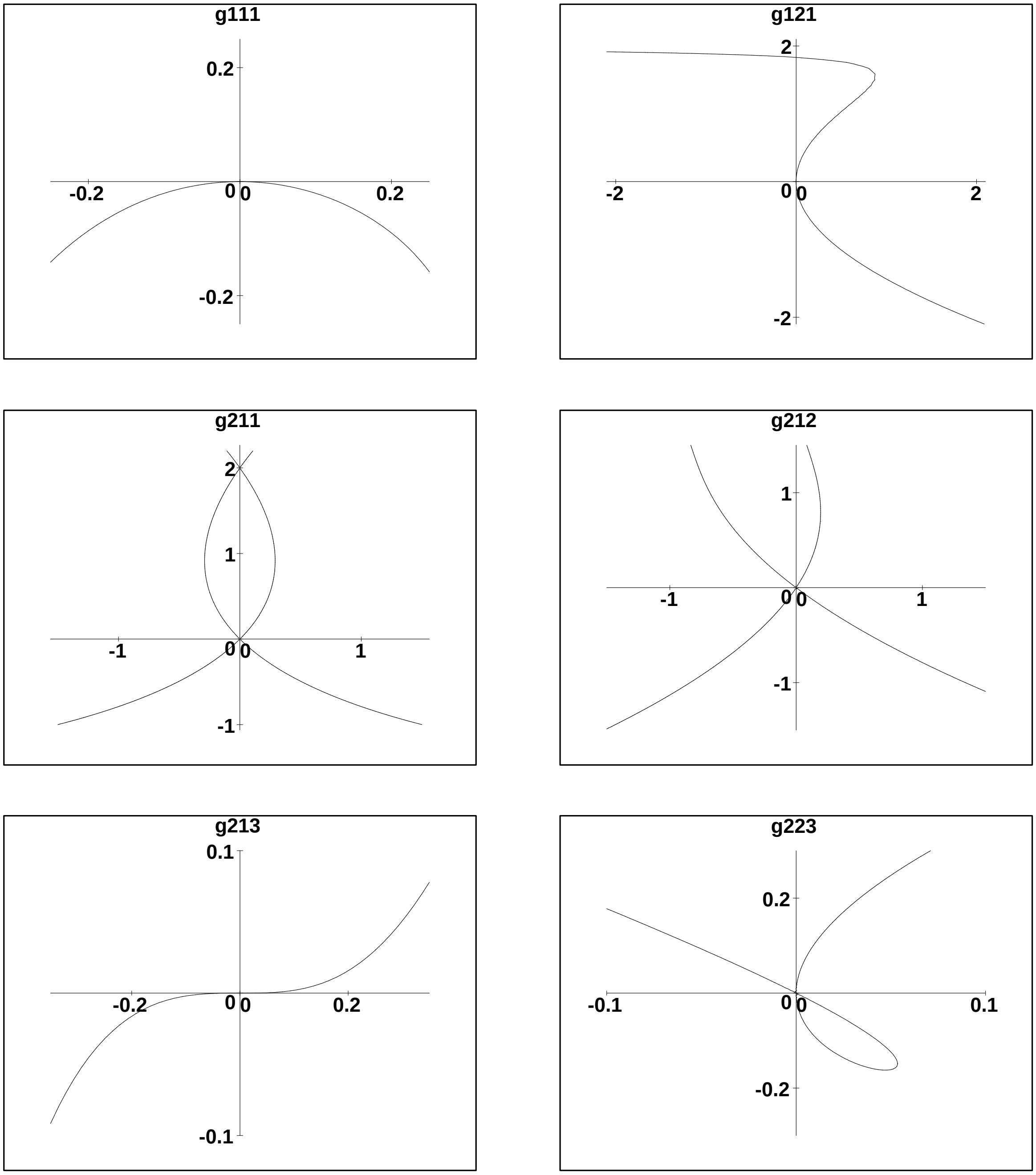

Figure 1: D_3-equivariant bifurcation problems: solutions within $\mathrm{Fix}(Z_2)$ are plotted x_1 versus λ, but conjugate solutions are omitted. From upper left to lower right the singularities g_{111}, g_{121}, g_{211}, g_{212}, g_{213}, and g_{223} are presented.

using the restrictions given by Theorem 8.2. The conditions yielded by $T(g,\vartheta) \subseteq t$ are $c_1(\lambda) - c_2(\lambda)p(b_2,b_1) = 0, -c_2(1)p(b_2,b_1) = 0$. They have two different solutions which have to be distinguished: $p(b_2,b_1) = 0, c_1(\lambda) = 0$ and $c_2(1) = 0, c_1(\lambda) = c_2(\lambda)p(b_2,b_1)$ with $p(b_2,b_1)$ arbitrary.

This gives two normal forms:

$$g_{121} = c_1(\sigma_2)b_1\sigma_2 + c_2(\sigma_1)b_2\sigma_1 + c_2(\sigma_2)b_2\sigma_2 + \epsilon_1 b_2\lambda + \epsilon_2 b_1\sigma_1 + \epsilon_3 b_1\lambda,$$

and $g_{122} = g_{111}$, where $\epsilon_i = \pm 1$. The first is visualized in Figure 1 for $c_2(\sigma_1) = 1, c_1(\sigma_2) = 0, c_2(\sigma_2) = -0.1, \epsilon_1 = -1, \epsilon_2 = 1, \epsilon_3 = -1$. g_{121} is equivalent to the normal form of codimension 2 in [12, p. 218]. Because we test for $T(g,\vartheta) \subseteq t$, but not for equality, it is possible that we derive a normal form of higher codimension than expected. In this case it fits nicely with the fact that the normal form in [12] has topological codimension 1. The second remark is that in Ansatz 1 and 2 $LT(rt)$ and $LT(q)$ generate the same $K[\lambda]$-module. However the results are different, since the introduction of parameters is different. It is a nice correspondence that g_{111} appears again under Ansatz 2.

3.) $LT(rt) = \{b_1\sigma_2, b_1\lambda^3, b_1\sigma_1, b_2\}$, $q = b_1\lambda$
This gives no new normal forms.

codim≥ 2

1.) $rt = \{b_1\sigma_2,\ b_1\sigma_1, b_1\lambda^2,\ b_2 + p(b_2, b_1\lambda)b_1\lambda + p(b_2,b_1)b_1\}$, $q = 0$

$$\begin{aligned} g_{211} &= c_1(\sigma_2)b_1\sigma_2 + c_2(\sigma_1)b_2\sigma_1 + c_2(\sigma_2)b_2\sigma_2 + \epsilon_1 b_2\lambda^2 + \epsilon_2 b_1\sigma_1 + \epsilon_3 b_1\lambda^2, \\ g_{212} &= c_1(\sigma_2)b_1\sigma_2 + c_2(\sigma_1)b_2\sigma_1 + c_2(\sigma_2)b_2\sigma_2 + \epsilon_1 b_2\lambda + \epsilon_2 b_1\sigma_1 + \epsilon_3 b_1\lambda^2, \\ g_{213} &= c_1(\sigma_1)b_1\sigma_1 + c_1(\sigma_2)b_1\sigma_2 + \epsilon_1 b_2 + \epsilon_2 b_1\lambda^3 + \epsilon_3 b_1\lambda\sigma_1, \end{aligned}$$

The Figure 1 refers to these bifurcation problems where the values $c_1(\sigma_2) = 1, c_2(\sigma_1) = 1, c_2(\sigma_2) = 1, \epsilon_1 = 1, \epsilon_2 = 1, \epsilon_3 = -1$ for g_{211}, $c_1(\sigma_2) = 1, c_2(\sigma_1) = 1, c_2(\sigma_2) = 1, \epsilon_1 = 1, \epsilon_2 = 1, \epsilon_3 = -1$ for g_{212}, and $c_1(\sigma_1) = 1, c_1(\sigma_2) = 1, \epsilon_1 = -1\epsilon_2 = -1, \epsilon_3 = 1$ for g_{213} was used.

2.) $rt = \{b_1\sigma_2,\ b_1\lambda^3,\ b_1\sigma_1 + p(b_1\sigma_1,\ b_1\lambda)b_1\lambda, b_2 + p(b_2, b_1\lambda)b_1\lambda + p(b_2,b_1)b_1\}$, $q = b_1\lambda^2$
This approach leads to 5 normal forms including those in Ansatz 1.) and the case codim ≥ 1 2. Ansatz g_{121}. The new one is

$$g_{223} = c_1(\sigma_2)b_1\sigma_2 + c_2(\sigma_1)b_2\sigma_1 + c_2(\sigma_2)b_2\sigma_2 + \epsilon_1 b_1\lambda\sigma_1 + \epsilon_2 b_2\lambda + \epsilon_3 b_1\lambda^2,$$

In Figure 1 the solutions in Fix(Z_2) are drawn for the values $c_1(\sigma_2) = 1, c_2(\sigma_1) = 1, c_2(\sigma_2) = 1, \epsilon_1 = -1, \epsilon_2 = -1, \epsilon_3 = 1$.

codim≥ 3

$rt = \{b_1\sigma_2,\ b_1\lambda^3,\ b_1\sigma_1 + p(b_1\sigma_1, b_1\lambda^2)b_1\lambda^2 + p(b_1\sigma 1, b_1\lambda)b_1\lambda,\ b_2 + p(b_2, b_1\lambda)b_1\lambda + p(b_2,b_1)b_1\}$, $q = 0$

We derive 3 normal forms which we have already considered and 4 new ones.

$$\begin{aligned} g_{313} &= c_1(\sigma_2)b_1\sigma_2 + c_2(\sigma_1)b_2\sigma_1 + c_2(\sigma_2)b_2\sigma_2 + \epsilon_1 b_2\lambda^3 \\ &\quad + \mathrm{sign}(c_1(\sigma_1))b_1\sigma_1 + \mathrm{sign}(c_2(\lambda^3)\cdot p(b_2,b_1))b_1\lambda^3, \\ g_{315} &= c_1(\sigma_2)b_1\sigma_2 + c_2(\sigma_1)b_2\sigma_1 + c_2(\sigma_2)b_2\sigma_2 \\ &\quad + \epsilon_1 b_1\lambda\sigma_1 + \mathrm{sign}(c_2(\lambda))b_2\lambda + \epsilon_3 b_1\lambda^3, \\ g_{316} &= c_1(\sigma_1)b_1\sigma_1 + c_1(\sigma_2)b_1\sigma_2 \\ &\quad + \mathrm{sign}(c_1(\lambda^4))b_1\lambda^4 + \mathrm{sign}(c_1(\lambda\sigma_1))b_1\lambda\sigma_1 + \mathrm{sign}(c_2(1))b_2, \\ g_{317} &= c\cdot b_2\sigma_1 + c_2(\sigma_2)b_2\sigma_2 + \epsilon_1 b_1\lambda^2 + \epsilon_2 b_2\lambda + \epsilon_3 b_1\lambda\sigma_1. \end{aligned}$$

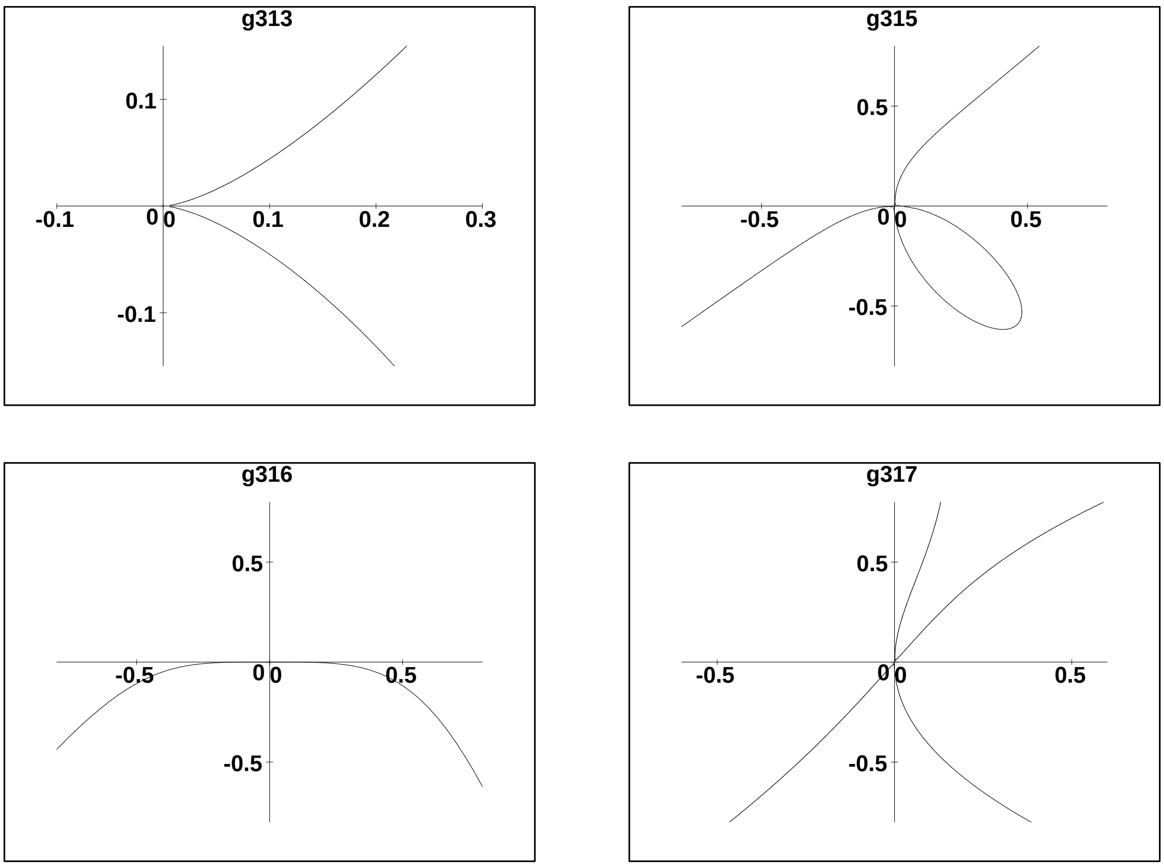

Figure 2: D_3-equivariant bifurcation problems: solutions within Fix(Z_2) are plotted x_1 versus λ, but conjugate solutions are omitted. From upper left to lower right the singularities g_{313}, g_{315}, g_{316}, and g_{317} are presented.

In Figure 2 the Z_2-invariant solutions are drawn for the parameter values $c_1(\sigma_2) = 1, c_2(\sigma_1) = 1, c_2(\sigma_2) = 1, \epsilon_1 = -1, \mathrm{sign}(c_1(\sigma_1)) = -1, \mathrm{sign}(c_2(\lambda^3) \cdot p(b_2, b_1)) = 1$ for g_{313}, $c_1(\sigma_2) = 1, c_2(\sigma_1) = 1, c_2(\sigma_2) = 1, \epsilon_1 = -1, \mathrm{sign}(c_2(\lambda)) = -1, \epsilon_3 = 1$ for g_{315}, $c_1(\sigma_1) = 1, c_1(\sigma_2) = 1, \mathrm{sign}(c_1(\lambda^4)) = -1, \mathrm{sign}(c_1(\lambda\sigma_1)) = -1, \mathrm{sign}(c_2(1)) = 1$ for g_{316}, and $c = 1, c_2(\sigma_2) = 1, \epsilon_1 = -1, \epsilon_2 = -1, \epsilon_3 = 1$ for g_{317}.

This list does not include the normal form of codimension 3 presented in [5]. Probably one will find it, if considerering more Ansatzes of codimension 3 and also try $RT(g, \vartheta) \subset rt, q \in t$ as requirement instead of $T(g, \vartheta) \subset t$.

10.2 Z_2

Although $Z_2 = \{id, s\}$ is the simplest group it is very important because this symmetry often appears in applications. We compute in invariant-equivariant coordinates $\sigma_1(x_1) = x_1^2, b_1(x_1) = x_1$. The weighted ordering is given by $b_1 > \sigma_1$ and $\deg b_1 = 1, \deg \sigma_1 = 2$. For completeness we mention that $S_1 = 1$ generates the module of commuting matrices.

The results are the following

codim≥ 0

$$g_{011} = \epsilon_1 b_1 \lambda + \epsilon_2 b_1 \sigma_1,$$

with $\epsilon_i = \pm 1$. In Figure 3 the solutions are drawn for $\epsilon_1 = 1, \epsilon_2 = -1$. This is the well-known pitchfork bifurcation.

codim≥ 1

$$\begin{aligned} g_{111} &= \epsilon_1 b_1 \lambda^2 + \epsilon_2 b_1 \sigma_1, \\ g_{121} &= \epsilon_1 b_1 \lambda + \epsilon_2 b_1 \sigma_1^2, \end{aligned}$$

The solutions are drawn in Figure 3 for $\epsilon_1 = 1, \epsilon_2 = -1$ (g_{111}) and $\epsilon_1 = 1 = \epsilon_2$ (g_{121}).

codim≥ 2

$$g_{211} = \epsilon_1 b_1 \lambda^2 + \epsilon_2 b_1 \sigma_1^2,$$

The solutions are drawn for $\epsilon_1 = 1, \epsilon_2 = -1$ in Figure 3.

$$g_{251} = \epsilon_1 b_1 \lambda^3 + \epsilon_2 b_1 \sigma_1,$$

In Figure 4 the solutions of g_{251} are plotted. $\epsilon_1 = 1, \epsilon_2 = -1$ is used again. The Ansatz $LT(rt) = \{b_1\sigma_1^2, b_1\lambda\sigma_1, b_1\lambda^2\}$ and $q = b_1\sigma_1$ leads to

$$\begin{aligned} g_{261} &= cb_1\sigma_1^2 + \epsilon_1 b_1 \lambda^3 + \epsilon_2 b_1 \lambda^2 \sigma_1, \\ g_{262} &= cb_1\sigma_1^2 + \epsilon_1 b_1 \lambda \sigma_1 + \epsilon_2 b_1 \lambda^2, \end{aligned}$$

The solutions of g_{261} are drawn for $\epsilon_1 = 1, \epsilon_2 = -1$ $c = 0.1$ and $c = 0$ in Figure 4. Note that for $c = 0$ the x_1-axis is a solution of $g_{261} = 0$. For g_{262} we have chosen $\epsilon_1 = 1, \epsilon_2 = -1, c = 1$. Observe that g_{262} is very similar to g_{211}. Probably, both are in the same contact equivalence class.

10.3 $Z_2 \times Z_2$

The group action of $Z_2 \times Z_2 = \{id, s_1, s_2, s_1s_2\}$ on $\mathbf{R}^2$ is given by

$$\vartheta(s_1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \vartheta(s_2) = \begin{pmatrix} 1 & 0 \\ 0 & 11 \end{pmatrix}.$$

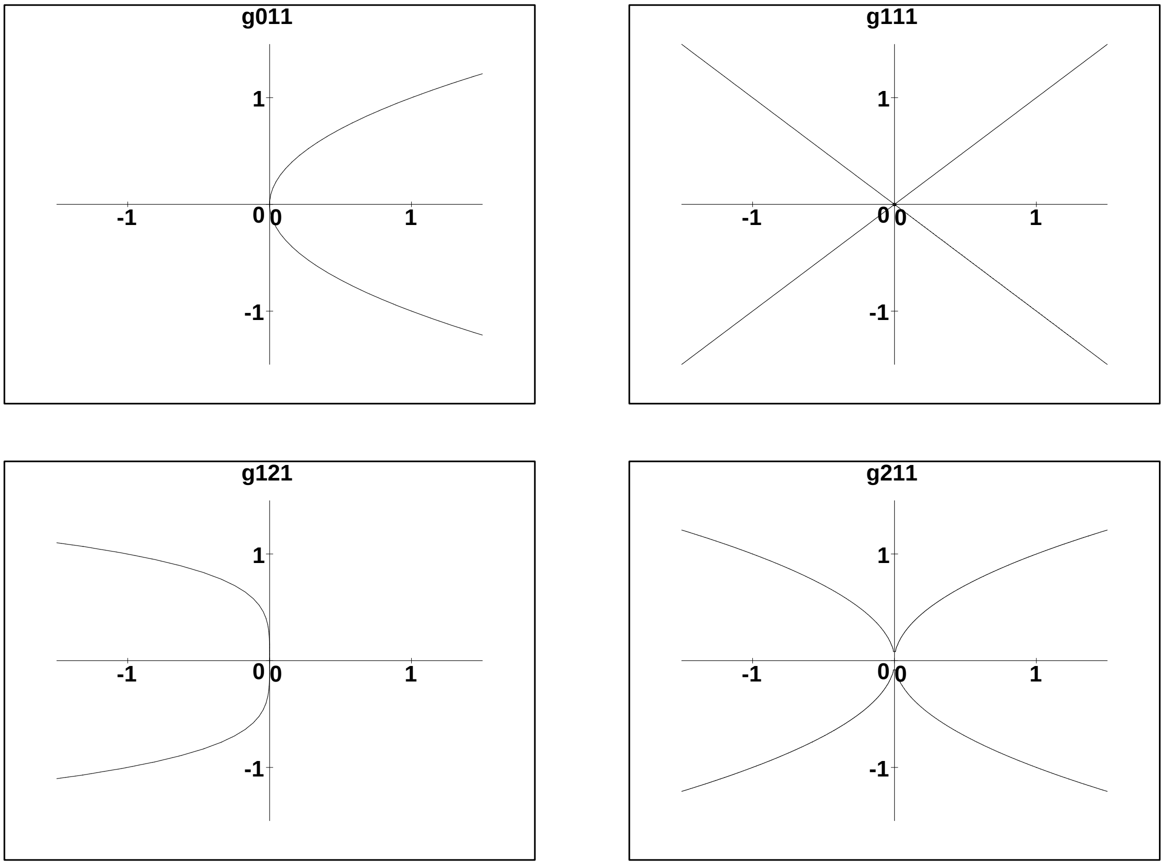

Figure 3: Z_2-equivariant bifurcation problems: solutions are plotted x_1 versus λ. From upper left to lower right the singularities g_{011}, g_{111}, g_{121}, g_{211} are presented.

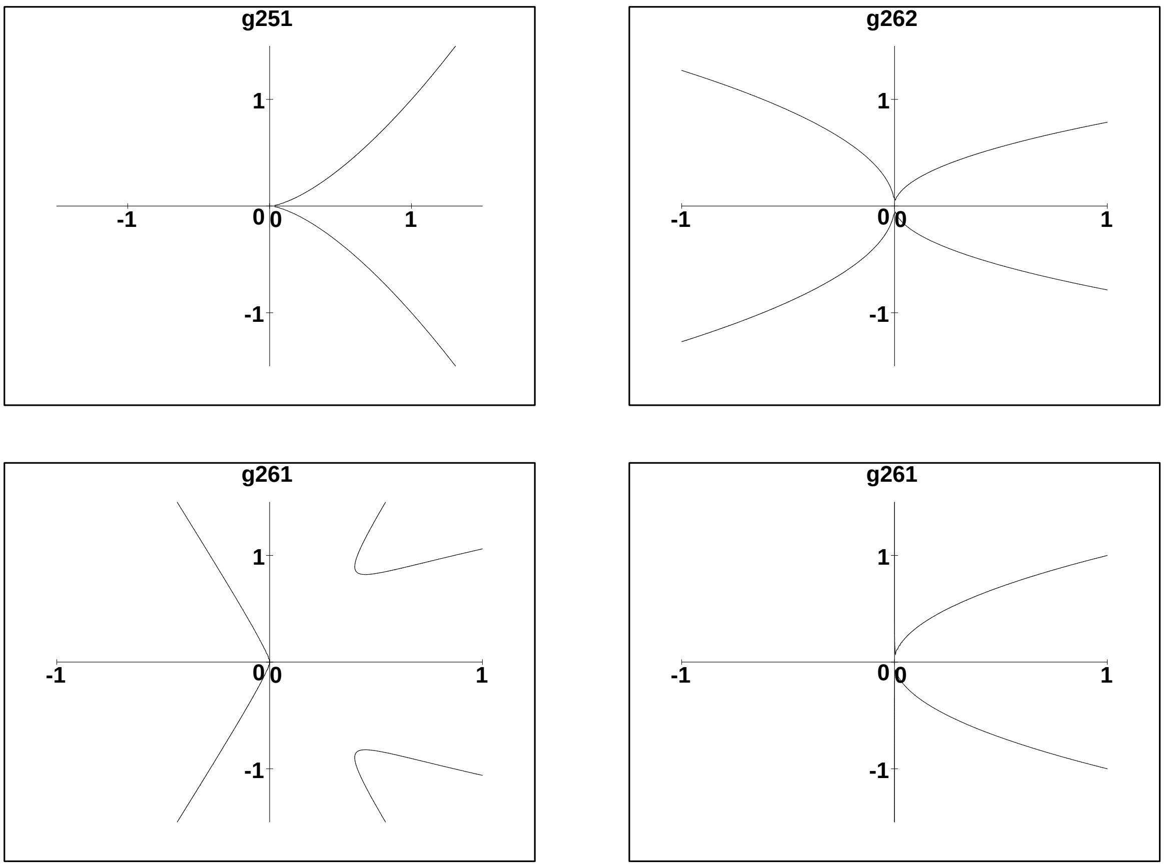

Figure 4: Z_2-equivariant bifurcation problems: solutions are plotted x_1 versus λ. From upper left to lower right the singularities g_{251}, g_{262}, and g_{261} for $c = 0.1$ and $c = 0$ are presented.

The invariants are $\sigma_1(x) = x_1^2, \sigma_2 = x_2^2$ and the equivariants are $b_1(x) = (x_1, 0), b_2(x) = (0, x_2)$. There are 4 fundamental commuting matrices.

There are two nontrivial subgroups $Z_2(s_1)$ and $Z_2(s_2)$. Considering the equivariant normal forms one observes that in each fixed point space $\mathrm{Fix}(Z_2(s_1))$ and $\mathrm{Fix}(Z_2(s_2))$ the same phenomena as described under subsection 10.2 and all combinations may occur. So our algorithm produces a lot of normal forms.

Especially interesting is the Ansatz

$LT(rt) = \{b_1\sigma_2, b_1\sigma_1, b_1\lambda, b_2\}$ and $t = rt$

One parameter is introduced and leads to the polynomial $b_2 + p(b_2, b_1)b_1 \in rt$. For each value of $p(b_2, b_1)$ the set rt is a Gröbner basis of a module with respect to the ordering $b_2 > b_1 > \sigma_2 > \sigma_1 > \lambda$ and weighting $(1, 1, 2, 2, 1)$. rt has codimension 1. A direct complement in $H_1(K[(\sigma_1, \sigma_2, \lambda), (b_1, b_2)])$ is generated by b_1.

The Ansatz for the normal form $g = \sum(c_1(\sigma_1^i\sigma_2^j\lambda^k)b_1 + c_2(\sigma_1^i\sigma_2^j\lambda^k)b_2)$ is chosen such that A is the complete space of polynomials of weighted degree 5. In order to search for bifurcation problems $c_1(1) = c_2(1) = 0$ is assumed.

The condition $RT(g, \vartheta) \subseteq rt$ implies the condition $c_1(\lambda) = c_2(\lambda) \cdot p(b_2.b_1)$. The resulting normal form after application of critmat, critequis,crittvar is

$$\begin{aligned} g \; = (\; & \mathrm{sign}(c_1(\sigma_1))\sigma_1 + c_1(\sigma_2)\sigma_2 + \mathrm{sign}(c_1(\lambda))\lambda)b_1 + \\ & (c_2(\sigma_1)\sigma_1 + \mathrm{sign}(c_2(\sigma_2))\sigma_2 + \mathrm{sign}(c_2(\lambda))\lambda)b_2 \end{aligned} \tag{35}$$

Here it is important to take into account that critmat and critequis are applied one after another and the problem in Example 9.12 appears. Applying critmat and critequis naively fixes the sign of $c_2(\sigma_1)$ and leaves $c_2(\sigma_2)$ unfixed. But this eliminates a lot of non-equivalent bifurcation problems. The normal form (35) is exactly the one discussed in [12], see as well [2].

$$g = (\epsilon_1\sigma_1 + m\sigma_2 + \epsilon_2\lambda)b_1 + (n\sigma_1 + \epsilon_3\sigma_2 + \epsilon_4\lambda)b_2$$

where the non-degeneracy conditions

$$m \neq \epsilon_2\epsilon_3\epsilon_4, \quad n \neq \epsilon_1\epsilon_2\epsilon_4, \quad mn \neq \epsilon\epsilon_3 \tag{36}$$

hold. We computed the 1-1-Gröbner basis of $RT(g, \vartheta)$ for arbitrary m, n, ϵ_i with respect to the ordering given by the variable order $b_2 > b_1 > \sigma_2 > \sigma_1 > \lambda$ and first weighting $(1, 1, 0, 0, 0)$ and second weighting $(1, 1, 2, 2, 1)$ and then lex.

```
                  2
groe := {b1*sig1 ,

          b1*sig1*lamb,

           - eps3*b2*sig2 + eps1*b1*sig1 + eps2*b1*lamb,

           - n*b2*sig1 - eps1*b1*sig1,

          m*b1*sig2 + eps1*b1*sig1 + eps2*b1*lamb,

                  2
          b1*lamb ,

          eps4*b2*lamb + eps2*b1*lamb}
```

This is only valid for the restrictions of the parameters $n \neq 0, \epsilon_4 \neq 0, \epsilon_1\epsilon_3 - mn \neq 0$, $3\epsilon_1\epsilon_3 - mn \neq 0, \epsilon_1 \neq 0, \epsilon_2\epsilon_3 - \epsilon_4 m \neq 0, \epsilon_2 \neq 0, 3\epsilon_1\epsilon_2\epsilon_3 - 2\epsilon_1\epsilon_4 m - \epsilon_2 mn \neq 0$. If one of these inequalities is violated the Gröbner basis looks different and the module $RT(g, \vartheta)$ might have a different codimension. Observe that the Buchberger algorithm can not make use of the information $\epsilon_i = \pm 1$. These conditions reflect nicely the non-degeneracy conditions (36) and other exceptional values where the solution structure of $g = 0$ is known to change qualitatively.

If the inequalities hold then $RT(g, \vartheta)$ has codimension 4 and a direct complement in $H_1(K[(\sigma_1, \sigma_2, \lambda), (b_1, b_2)])$ is generated by $b_1, b_2, b_1\lambda, b_1\sigma_1$. Since $g_\lambda = b_1\epsilon_2 + b_2\epsilon_4$ completes the tangent space, the normal form has codimension ≤ 3. By Proposition 2.3, p.424 in [11] it has exactly algebraic codimension 3. The topological codimension is 1 and m, n are moduli parameters. Thus an universal unfolding is $g + \alpha_1 b_1 + \alpha_2 b_1 \lambda + \alpha_3 b_1 \sigma_1$.

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