Duality by Reproducing Kernels

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Abstract

Let A be a determined or overdetermined elliptic differential operator on a smooth compact manifold X. Write $\mathcal{S}_A(\mathcal{D})$ for the space of solutions to the system $Au = 0$ in a domain $\mathcal{D} \Subset X$. Using reproducing kernels related to various Hilbert structures on subspaces of $\mathcal{S}_A(\mathcal{D})$ we show explicit identifications of the dual spaces. To prove the "regularity" of reproducing kernels up to the boundary of $\mathcal{D}$ we specify them as resolution operators of abstract Neumann problems. The matter thus reduces to a regularity theorem for the Neumann problem, a well-known example being the $\bar{\partial}$-Neumann problem. The duality itself takes place only for those domains $\mathcal{D}$ which possess certain convexity properties with respect to A.

Contents

1   Introduction

The present work continues our paper [NST98] and grows out of a desire to highlight the link of duality theorems for solutions of elliptic equations to basic problems of partial differential equations, such as the existence and regularity of solutions.

Let

$$V^0 \xrightarrow{\;T\;} V^1,$$
$$V^0 \xleftarrow{\;S\;} V^1$$

be two continuous mappings of Fréchet spaces, such that $ST = I$ on a closed subspace U^0 of V^0. In other words, the identity mapping of U^0 factors through T, hence the restriction of T to U^0 is one-to-one and the image of U^0 under T is a closed subspace of V^1.

Obviously, S maps TU^0 to U^0. If $\mathcal{F}$ is a continuous linear functional on U^0 then

$$\langle \mathcal{F}, u \rangle = \langle S'\mathcal{F}, Tu \rangle$$

for any $u \in U^0$, where S' is the transpose of $S : TU^0 \to U^0$. Moreover, $S'\mathcal{F} = 0$ implies $\mathcal{F} = 0$. We thus obtain an one-to-one mapping $(U^0)' \to (TU^0)'$ given by $\mathcal{F} \mapsto S'\mathcal{F}$. The problem of identifying the dual of U^0 reduces to the description of the range of S'.

We restrict our attention to the case where both V^0 and V^1 are function spaces and U^0 is a space of solutions to some elliptic equation $Au = 0$. Even for explicitly given S the range of S' cannot be described by mere tools of partial differential equations. The crucial fact is that by the uniqueness theorem C^∞ functions with compact support are not dense in U^0, and so the functionals of $(U^0)'$ cannot be specified within distributions. To handle S' one therefore needs much more refined analysis. For a deeper discussion we refer the reader to [Kha66] and Ch. 3 in [Tar97].

It is usually the case for hypoelliptic equations that U^0 is in fact a nuclear space. By the Schwartz kernel theorem, the mapping $S : TU^0 \to U^0$ has a kernel $K_S \in U^0 \hat{\otimes}_\pi (TU^0)'$ (cf. 1.4.1 in [Tar95a]). We call K_S a *reproducing kernel*, for

$$u(x) = \langle K_S(x, \cdot), Tu \rangle \tag{1.1}$$

for all x in the domain of $u \in U^0$.

The advantage of using reproducing kernels lies in the fact that it enables us to write $S'\mathcal{F} = \langle \mathcal{F}, K_S(\cdot, y) \rangle$. The right-hand side here is called the *indicatrix* of the functional $\mathcal{F}$.

This concept was first studied in the particular cases of holomorphic and harmonic functions on certain explicitly given domains, cf. [Mar63, Aiz66, Zna79]. However, representing analytic functionals as analytic functions on

domains in $\mathbb{C}^n$ requires specific tools of the general theory of partial differential equations, cf. [Zor82, Sto95].

Our main result consists of the following. The correspondence $\mathcal{F} \mapsto S'\mathcal{F}$ maps the dual space of U^0 to the space of solutions to $Av = 0$ in the same domain, which grow near the boundary in a sense dual to the growth of solutions in U^0. This mapping is always one-to-one but not necessarily onto. Its surjectivity is equivalent to the regularity of certain projection onto the space of solutions.

We evaluate the projection through the resolution operator of a generalised Neumann problem related to A. The desired regularity of the projection just amounts to that of the solution to the Neumann problem. We thus bring together two different areas of analysis in which the problem of regularity turns out to be of key importance.

2 A general scheme

2.1 Spaces of solutions to elliptic systems

Let X be a C^∞ manifold of dimension n with a smooth boundary ∂X. The case $\partial X = \emptyset$ is also included. We tacitly assume that X is embedded into a smooth closed manifold $\tilde{X}$ of the same dimension.

For any smooth $\mathbb{C}$-vector bundles E and F over X, we write $\mathrm{Diff}^m(X; E, F)$ for the space of all linear partial differential operators of order $\leq m$ between sections of E and F.

Denote by E^* the conjugate bundle of E. Any Hermitean metric $(\cdot, \cdot)_x$ on E gives rise to a sesquilinear bundle isomorphism $*_E \colon E \to E^*$ by the equality $\langle *_E v, u \rangle_x = (u, v)_x$ for all sections u and v of E.

We pick a volume form dx on X, thus identifying the dual and conjugate bundles. For $A \in \mathrm{Diff}^m(X; E, F)$, denote by $A' \in \mathrm{Diff}^m(X; F^*, E^*)$ the transposed operator and by $A^* \in \mathrm{Diff}^m(X; F, E)$ the formal adjoint operator. We obviously have

$$A^* = *_E^{-1} A' *_F,$$

cf. [Tar95b, 4.1.4] and elsewhere.

Write $\sigma^m(A)$ for the principal homogeneous symbol of order m of the operator A, $\sigma^m(A)$ living on the cotangent bundle T^*X of X. From now on we assume that $\sigma^m(A)$ is injective away from the zero section of T^*X. Hence it follows that the Laplacian $\Delta = A^*A$ is an elliptic differential operator of order $2m$ on X.

Given any open set U in $\overset{\circ}{X}$, the interior of X, let $\mathcal{S}_A(U)$ stand for the space of solutions to the equation $Au = 0$ in U with the topology of uniform

convergence on compact subsets of U. It is known that $\mathcal{S}_A(U)$ is a Fréchet-Schwartz space.

Denote by $\mathcal{S}_A(U)'$ the dual space of $\mathcal{S}_A(U)$, i.e., the space of all continuous linear functionals on $\mathcal{S}_A(U)$. As usual, we give $\mathcal{S}_A(U)'$ the strong topology, i.e., the topology of uniform convergence of functionals on bounded subsets of $\mathcal{S}_A(U)$.

Throughout this paper we assume that the Laplacian Δ possesses the following Unique Continuation Property:

$(U)_s$ Given any domain $\mathcal{D} \subset \overset{\circ}{X}$, if $u \in \mathcal{S}_\Delta(\mathcal{D})$ vanishes on a nonempty open subset of $\mathcal{D}$ then $u \equiv 0$ in $\mathcal{D}$.

This property implies in particular the existence of a two-sided fundamental solution for Δ in the interior of X.

Natural domains for solutions to $Au = 0$ are certainly open subsets of the interior of X. However, some problems require to consider solutions on sets σ in X which are not open. Here we are interested not simply in restrictions of solutions to the given set, but also in the local solutions of the system $Au = 0$ on σ. By these we mean solutions to the system on various neighbourhoods of σ depending on the solution.

If σ is a closed subset of X, then $\mathcal{S}_A(\sigma)$ stands for the space of (equivalence classes of) local solutions to $Au = 0$ on σ. Two such solutions are equivalent if there is a neighbourhood of σ where they are equal. In $\mathcal{S}_A(\sigma)$, a sequence $\{u_\nu\}$ is said to converge if there exists a neighbourhood $\mathcal{N}$ of σ such that all the solutions are defined at least in $\mathcal{N}$ and converge uniformly on compact subsets of $\mathcal{N}$.

Alternatively the space $\mathcal{S}_A(\sigma)$ can be described as the inductive limit of spaces $\mathcal{S}_A(U_\nu)$, where $\{U_\nu\}$ is any decreasing sequence of open sets containing σ, such that each neighbourhood of σ contains some U_ν, and such that each connected component of each U_ν intersects σ. This latter condition guarantees that the mappings $\mathcal{S}_A(U_\nu) \to \mathcal{S}_A(\sigma)$ are one-to-one. Then $\mathcal{S}_A(\sigma)$ is necessarily a Hausdorff space.

For an open set $U \subset X$, we denote by $L^2(U, E)$ the Hilbert space of all square integrable sections of E over U with scalar product

$$(u, v)_{L^2(U,E)} = \int_U (u, v)_x dx.$$

More generally, write $H^s(U, E)$, $s \in \mathbb{Z}_+$, for the Sobolev space of sections of E over U, whose weak derivatives up to order s belong to $L^2(U, E)$. We define $H^s(U, E)$ with $s = -1, -2, \ldots$ to be the dual space for $H^{-s}(U, E)$ with respect to the $L^2(U, E)$-pairing.

We also denote by $\mathcal{S}_A^{(s)}(U)$, with any integer s, the closed subspace of $H^s(U, E)$ consisting of all weak solutions to $Au = 0$ in U. It is well known

that $\mathcal{S}_A^{(s)}(U)$ is a separable Hilbert space with reproducing kernel, cf [Tar95a] and elsewhere.

The union of the spaces $\mathcal{S}_A^{(s)}(U)$ over all $s \in \mathbb{Z}$ is perhaps of particular interest. For regular U it consists of all solutions to $Au = 0$ in U, which are of finite order of growth near the boundary of U. This means that u fulfills an estimate

$$|u(x)| \leq \frac{C}{\text{dist}(x, \partial U)^N}$$

for all $x \in U$, with N and C constants depending on u.

Write $\mathcal{S}_A^{(f)}(U)$ for the space of solutions to $Au = 0$ in U which have a finite order of growth near ∂U. By the above, we can give $\mathcal{S}_A^{(f)}(U)$ the inductive limit topology of the sequence $\mathcal{S}_A^{(-s)}(U)$, $s \in \mathbb{N}$.

Since the Dirichlet problem for the Laplacian $\Delta = A^*A$ in U is uniquely solvable, the topology of $\mathcal{S}_\Delta^{(f)}(U)$ can be equivalently described in the following way. Pick a Dirichlet system $u \mapsto t(u)$ of order $m - 1$ on the boundary of U, provided the latter is smooth. By [ST95], for each $u \in \mathcal{S}_\Delta^{(f)}(U)$ the Dirichlet data $t(u)$ are well defined in

$$\bigoplus_{j=0}^{m-1} \mathcal{D}'(\partial U, F_j),$$

F_j being some vector bundles in a neighbourhood of ∂U.

Lemma 2.1 *A sequence $\{u_\nu\}$ converges to u in $\mathcal{S}_\Delta^{(f)}(U)$ if and only if $t(u_\nu) \to t(u)$ in $\oplus_{j=0}^{m-1} \mathcal{D}'(\partial U, F_j)$.*

Proof. Cf. Theorem 2.32.

$\square$

2.2 Duality

Let Σ_1 be a vector subspace of $\mathcal{S}_A(U)$ endowed with topology τ_1 which is not weaker then the Fréchet-Schwartz topology of $\mathcal{S}_A(U)$. Denote by Σ_1' the dual space of Σ_1.

Suppose V is a separable Hilbert space of functions in a domain U with a scalar product $h(\cdot, \cdot)$. Let moreover there be a topological vector space Σ_2, continuous linear mappings

$$\Sigma_2 \overset{i_2}{\hookrightarrow} V \overset{i_1}{\hookrightarrow} \Sigma_1$$

and a sesquilinear pairing

$$\tilde{h}(\cdot, \cdot): \ \Sigma_1 \times \Sigma_2 \to \mathbb{C}$$

such that

1) $\tilde{h}(\cdot, \cdot)$ is separately continuous;

2) $\tilde{h}(i_1 u, v) = h(u, i_2 v)$ for all $u \in V$ and $v \in \Sigma_2$.

Under these assumptions the pairing $\tilde{h}(\cdot, \cdot)$ induces a continuous mapping $\mathcal{J} : \Sigma_2 \to \Sigma_1'$ by

$$
\begin{aligned}
(\mathcal{J}v)(\cdot) &:= \mathcal{F}_v(\cdot) \\
&:= \tilde{h}(\cdot, v)
\end{aligned}
\tag{2.1}
$$

for any $v \in \Sigma_2$.

Lemma 2.2 *If $i_2 : \Sigma_2 \to V$ is one-to-one then the mapping $\mathcal{J}$ given by (2.1) is one-to-one.*

Proof. If $\mathcal{F}_v(\cdot)$ is identically zero then

$$
\begin{aligned}
\mathcal{F}_v(i_1 i_2 v) &= h(i_2 v, i_2 v) \\
&= 0.
\end{aligned}
$$

As $h(\cdot, \cdot)$ is a scalar product on V we conclude that $i_2 v = 0$. Hence $v = 0$ if i_2 is one-to-one.

$\square$

By a priori estimates for solutions of elliptic systems it is easy to see that the inclusion

$$
i = i_1 : \ V \to \mathcal{S}_A(U) =: \Sigma_1
$$

is continuous if and only if all the evaluation functionals $x \mapsto u(x)$, $x \in U$, are continuous on V. This latter just amounts to saying that the Hilbert space V has a reproducing kernel $K(x, y) \in V' \otimes V$, i.e., any $u \in V$ represents by the formula

$$
u(x) = h(u, K(x, \cdot))
$$

for all $x \in U$, cf. [Aro50].

Define $*_V : V \to V'$ by $\langle *_V v, u \rangle = h(u, v)$ for all $u \in V$. By the theorem of Riesz, $*_V$ is a sesquilinear isomorphism of V onto V'.

Theorem 2.3 *Let V be a Hilbert space with reproducing kernel $K(\cdot, \cdot)$. Suppose i_2 is one-to-one. Then the mapping $\mathcal{J}$ given by (2.1) is onto if and only if*

1) $i_1 i_2(\Sigma_2)$ *is dense in Σ_1;*

2) *for every $\mathcal{F} \in \Sigma_1'$, the section $x \mapsto *_V^{-1} \langle \mathcal{F}, i_1 K(x, \cdot) \rangle$ belongs to $i_2(\Sigma_2)$.*

Note that
$$h\left(u(y), *_V w(x) \otimes v(y)\right) = h(u, v)\, w(x)$$
for each $x \in U$ and $u, v, w \in V$, as is easy to check.

Proof.

Necessity. Let $\mathcal{F}$ be a continuous linear functional on Σ_1 vanishing on $i_1 i_2(\Sigma_2)$. By the Hahn-Banach Theorem, we will prove that $i_1 i_2(\Sigma_2)$ is dense in Σ_1 once we show that $\mathcal{F} \equiv 0$.

By assumption, there is an element $v \in \Sigma_2$ such that $\mathcal{F}_v = \mathcal{F}$. It follows that

$$
\begin{aligned}
\mathcal{F}_v\big(i_1 i_2 v\big) &= h\big(i_2 v, i_2 v\big) \\
&= 0,
\end{aligned}
$$

and so $v = 0$. Hence $\mathcal{F} \equiv 0$, as desired.

Further, an easy calculation shows that

$$
\begin{aligned}
\langle \mathcal{F}, i_1 K(x, \cdot) \rangle &= \tilde{h}\left(i_1 K(x, \cdot), v\right) \\
&= h\left(K(x, \cdot), i_2 v\right) \\
&= *_V h\left(i_2 v, K(x, \cdot)\right) \\
&= *_V \left(i_2 v\right)(x) \\
&\in *_V i_2(\Sigma_2),
\end{aligned}
$$

the fourth equality being due to the fact that $K(\cdot, \cdot)$ is a reproducing kernel of V. This proves the necessity.

Sufficiency. Let conditions 1) and 2) of the theorem hold. The task is now to show that the mapping $\mathcal{J} : \Sigma_2 \to \Sigma_1'$ is onto.

Lemma 2.4 *Let $u \in \Sigma_1$. Then the formula*

$$u(x) = \tilde{h}\left(u, i_2^{-1} i_1\, K(x, \cdot)\right). \tag{2.2}$$

is valid for all $x \in U$.

Proof. Indeed, by a priori estimates for elliptic systems all evaluation functionals $\delta_x(u) = u(x)$, $x \in U$, are continuous on Σ_1. The condition 2) then implies that

$$
\begin{aligned}
*_V^{-1} \langle \delta_x, i_1 K(y, \cdot) \rangle &= i_1 *_V^{-1} K(y, x) \\
&= i_1 K(x, y) \\
&\in E_x^* \otimes i_2(\Sigma_2)
\end{aligned}
$$

for every fixed $x \in U$. It follows that the pairing $\tilde{h}\left(u, i_2^{-1} i_1 K(x, \cdot)\right)$ is well defined.

Pick a sequence $\{u_\nu\}$ in Σ_2 such that $\{i_1 i_2 u_\nu\}$ approximates u in Σ_1. Since $K(\cdot, \cdot)$ is a reproducing kernel we see that

$$(i_2 u_\nu)(x) = h(i_2 u_\nu, K(x, \cdot))$$

whence

$$
\begin{aligned}
(i_1 i_2 u_\nu)(x) &= h\left(i_2 u_\nu, i_2 i_2^{-1} i_{1,x} K(x, \cdot)\right) \\
&= \tilde{h}\left(i_1 i_2 u_\nu, i_2^{-1} i_{1,x} K(x, \cdot)\right)
\end{aligned}
\tag{2.3}
$$

for all $x \in U$ and $\nu = 1, 2, \ldots$. Since the pairing $\tilde{h}(\cdot, \cdot)$ is separately continuous, the passage to the limit in (2.3), when $\nu \to \infty$, yields (2.2). The lemma is proved.

$\square$

We are now in a position to complete the proof of Theorem 2.3. Suppose $\mathcal{F} \in \Sigma_1'$. Then by Lemma 2.4 we see that

$$
\begin{aligned}
\mathcal{F}(u) &= \mathcal{F}\left(\tilde{h}\left(u, i_2^{-1} i_1\, K(x, \cdot)\right)\right) \\
&= \tilde{h}(u, v)
\end{aligned}
$$

where

$$
\begin{aligned}
v &= i_2^{-1} \langle \overline{\mathcal{F}}, i_1 K(\cdot, y) \rangle \\
&= i_2^{-1} *_V^{-1} \langle \overline{\mathcal{F}}, i_1 K(y, \cdot) \rangle \\
&\in \Sigma_2.
\end{aligned}
$$

The last reasoning is an immediate consequence of condition 2), thus showing the theorem.

$\square$

Corollary 2.5 *If Σ_1 is a closed subspace of $\mathcal{S}_A(U)$ then the condition 2) of Theorem 2.3 is equivalent to the following one:*

2′) *for each fixed $y \in U$, the section $i_{1,y} *_V^{-1} K(\cdot, y)$ belongs to $i_2(\Sigma_2) \otimes E_y$.*

Proof. That 2) implies 2′) we have already established in the proof of Lemma 2.4. It remains to show the implication 2′) $\Rightarrow$ 2).

Pick a continuous linear functional $\mathcal{F}$ on Σ_1. Since Σ_1 is a closed subspace of $C_{\mathrm{loc}}(U, E)$, the space of continuous sections of E over U, this functional extends, by the Hahn-Banach Theorem, to an E^*-valued measure m with a compact support in U. For any $x \in U$,

$$
\begin{aligned}
*_V^{-1} \langle \mathcal{F}, i_1 K(x, \cdot) \rangle &= \int_{\mathrm{supp}\, m} \left\langle i_{1,y} *_V^{-1} K(x, y), dm(y) \right\rangle_y \\
&\in i_2(\Sigma_2),
\end{aligned}
$$

since i_1 is continuous and $\operatorname{supp} m$ is a compact subset of U. This completes the proof.

$\square$

Corollary 2.6 *Let i_2 be one-to-one. Suppose the Closed Graph Theorem is valid for mappings between Σ_2 and Σ_1'. Then the mapping $\mathcal{J} : \Sigma_2 \to \Sigma_1'$ defined by (2.1) is a topological isomorphism between these spaces if and only if the conditions 1) and 2) of Theorem 2.3 hold.*

Proof. This follows from the continuity of the mapping $\mathcal{J}$ and the Closed Graph Theorem.

$\square$

Let $\mathfrak{S}_1$, $\mathfrak{S}_2$ and $\mathfrak{V}$ be closed subspaces of the spaces Σ_1, Σ_2 and V, respectively. We thus get a commutative diagram

$$
\begin{array}{ccccc}
\Sigma_2 & \overset{i_2}{\hookrightarrow} & V & \overset{i_1}{\hookrightarrow} & \Sigma_1 \\
\cup & & \cup & & \cup \\
\mathfrak{S}_2 & \overset{i_2}{\hookrightarrow} & \mathfrak{V} & \overset{i_1}{\hookrightarrow} & \mathfrak{S}_1.
\end{array}
\tag{2.4}
$$

Once again the pairing $\tilde{h}(\cdot, \cdot)$ induces the mapping $\mathfrak{J} : \mathfrak{S}_2 \to \mathfrak{S}_1'$ which is to certain extent the restriction of $\mathcal{J}$ to $\mathfrak{S}_2$. We tacitly assume that the continuous mappings under study are characterised in terms of convergent sequences and that the Closed Graph Theorem is valid for mappings between Σ_2 and Σ_1'.

Write $\pi : V \to \mathfrak{V}$ for the corresponding orthogonal projection.

Corollary 2.7 *Let i_2 be one-to-one. Suppose $\mathcal{J}$ is a topological isomorphism of Σ_2 onto Σ_1'. Then the mapping $\mathfrak{J}$ is a topological isomorphism of $\mathfrak{S}_2$ onto $\mathfrak{S}_1'$ if and only if*

1) $i_1 i_2 \mathfrak{S}_2$ *is dense in $\mathfrak{S}_1$;*
2) *the projection π maps $i_2(\Sigma_2)$ continuously into $i_2(\mathfrak{S}_2)$.*

Proof. Since i_2 is one-to-one, so is the restriction of i_2 to $\mathfrak{S}_2$, too. Hence the mapping $\mathfrak{J}$ is one-to-one, by Lemma 2.2. It remains to prove that conditions 1) and 2) of Theorem 2.3, if applied to $\mathfrak{S}_1$, $\mathfrak{S}_2$ and $\mathfrak{V}$, are equivalent to conditions 1) and 2) of the present corollary. Of course, the conditions labelled by 1) coincide. Thus, we restrict our attention to the conditions labelled by 2).

Necessity. Pick a sequence $\{v_\nu\}$ converging in Σ_2 to a limit v. Then the corresponding sequence of functionals $\mathcal{F}_{v_\nu}$ converges to $\mathcal{F}_v$ in Σ_1'. Clearly, the restrictions of $\mathcal{F}_{v_\nu}$ to $\mathfrak{S}_1$ converge in turn to the restriction of $\mathcal{F}_v$ to $\mathfrak{S}_1$ in the dual space $\mathfrak{S}_1'$. If the mapping $\mathfrak{J}$ is a topological isomorphism of $\mathfrak{S}_2$ onto $\mathfrak{S}_1'$

then there exists a sequence $\mathfrak{v}_\nu \in \mathfrak{S}_2$ converging to a limit $\mathfrak{v}$ in this space, such that

$$
\begin{aligned}
\mathcal{F}_{v_\nu}(\mathfrak{u}) &= \tilde{h}(\mathfrak{u}, \mathfrak{v}_\nu), \\
\mathcal{F}_v(\mathfrak{u}) &= \tilde{h}(\mathfrak{u}, \mathfrak{v})
\end{aligned}
$$

for all $\mathfrak{u} \in \mathfrak{S}_1$. In particular, for all $\mathfrak{u} \in \mathfrak{V}$ we get

$$
\begin{aligned}
h(\mathfrak{u}, \pi i_2 v_\nu) &= h(\pi \mathfrak{u}, i_2 v_\nu) \\
&= h(\mathfrak{u}, i_2 v_\nu) \\
&= \tilde{h}(i_1 \mathfrak{u}, v_\nu) \\
&= \mathcal{F}_{v_\nu}(i_1 \mathfrak{u}) \\
&= \tilde{h}(i_1 \mathfrak{u}, \mathfrak{v}_\nu) \\
&= h(\mathfrak{u}, i_2 \mathfrak{v}_\nu),
\end{aligned}
$$

i.e., $\pi i_2 v_\nu = i_2 \mathfrak{v}_\nu$ and the projection π maps $i_2(\Sigma_2)$ continuously into $i_2(\mathfrak{S}_2)$, as desired.

Sufficiency. Conversely, let conditions 1) and 2) of Corollary 2.7 hold. Pick an orthonormal basis $\{b_\nu\}$ in $\mathfrak{V}$. It is well known that the reproducing kernel of $\mathfrak{V}$ is given in the form

$$
\mathfrak{K}(x, y) = \sum_\nu *_{\mathfrak{V}} b_\nu(x) \otimes b_\nu(y).
$$

As every orthonormal basis in $\mathfrak{V}$ can be extended to an orthonormal basis in V we see that

$$
\mathfrak{K}(x, y) = \pi_y K(x, y)
$$

for all x and y. By Theorem 2.3, the section $x \mapsto *_V^{-1} \langle \mathcal{F}, i_1 K(x, \cdot) \rangle$ belongs to $i_2(\Sigma_2)$ for all $\mathcal{F} \in \Sigma_1'$. By the Hahn-Banach Theorem every functional $\mathfrak{F} \in \mathfrak{S}_1'$ actually extends continuously to a functional $\mathcal{F} \in \Sigma_1'$. Hence condition 2) yields

$$
\begin{aligned}
*_V^{-1} \langle \mathfrak{F}, i_1 \mathfrak{K}(x, \cdot) \rangle &= *_V^{-1} \langle \mathcal{F}, i_1 \mathfrak{K}(x, \cdot) \rangle \\
&= \pi_x *_V^{-1} \langle \mathcal{F}, i_1 K(x, \cdot) \rangle \\
&\in i_2(\mathfrak{S}_2),
\end{aligned}
$$

the latter inclusion being due to the commutative diagram (2.4). We thus conclude that the condition 2) of Theorem 2.3 is fulfilled for $\mathfrak{J}$, and that this mapping is onto. The topological arguments now follow from the Closed Graph Theorem.

$\square$

Note that if $\dim \Sigma_1 < \infty$ then conditions 1) and 2) of Theorem 2.3 imply that $i_1 i_2(\Sigma_2) = \Sigma_1$ and $\dim \Sigma_1 = \dim \Sigma_2$. Conversely, suppose these latter

conditions $i_1 i_2(\Sigma_2) = \Sigma_1$ and $\dim \Sigma_1 = \dim \Sigma_2$ are fulfilled. Hence it follows that $i_2(\Sigma_2)$ is a closed subspace of V. When replacing V by $i_2(\Sigma_2)$ and i_1 by its restriction to $i_2(\Sigma_2)$, we still have the same mapping $\mathcal{J}$. The reproducing kernel $K(\cdot, \cdot)$ is given by

$$
K(x, y) = \sum_{\nu=1}^{\dim \Sigma_1} (i_2 b_\nu)^* (x) \otimes (i_2 b_\nu) (y)
$$

where $\{b_\nu\}$ is a basis in Σ_2 with the property that $\{i_2 b_\nu\}$ is an orthonormal basis in $i_2(\Sigma_2)$. Given any $\mathcal{F} \in \Sigma_1'$, we get

$$
\begin{aligned}
*_V^{-1} \langle \mathcal{F}, i_1 K(x, \cdot) \rangle &= \sum_{\nu=1}^{\dim \Sigma_1} \overline{\mathcal{F}(i_1 i_2 b_\nu)} (i_2 b_\nu) (x) \\
&\in i_2(\Sigma_2).
\end{aligned}
$$

Similar considerations apply to the commutative diagram (2.4). In this setting the projection π always maps V continuously into $i_2(\mathfrak{S}_2)$.

How can we derive the necessary information on the projection π under general assumptions? In many cases it can be obtained from a Neumann problem.

2.3   Neumann problem

In our applications $\mathfrak{V}$ is usually a Hilbert space of solutions to the equation $Au = 0$ in a domain $U \subset X$.

Let the operator A be included into an elliptic compatibility complex of differential operators $A^i \in \mathrm{Diff}^{m_i}(X; E^i, E^{i+1})$, $i = 0, 1, \ldots, N$, over X, with $A^0 = A$.

Suppose that $V^i \hookrightarrow \mathcal{D}'(\mathring{X}, E^i)$, $i = 0, 1, \ldots, N$, are Hilbert spaces of sections of E^i over X, such that

1) $V^i \cap C^\infty(X, E^i)$ is dense in V^i for all $i = 0, 1, \ldots, N$;

2) A^i maps $V^i \cap C^\infty(X, E^i)$ to $V^{i+1} \cap C^\infty(X, E^{i+1})$.

Let $\mathcal{D}_T^i$ be the set of all sections $u \in V^i$, for which there is a sequence $\{u_\nu\}$ with the following properties:

1) $u_\nu \in V^i \cap C^\infty(X, E^i)$;

2) $\{u_\nu\}$ converges to u in V^i; and

3) $\{A u_\nu\}$ is a Cauchy sequence in V^{i+1}.

The mapping $T : \mathcal{D}_T^i \to V^{i+1}$ defined by $Tu = \lim Au_\nu$, where $\{u_\nu\}$ is a sequence with properties 1)–3), is called the maximal operator generated by A.

Note that T is well defined. Indeed, if $\{u_\nu'\}$ is another sequence satisfying 1)–3), and $f = \lim Au_\nu'$, then for all $g \in C^\infty(X, E^{i+1*})$ with a compact support in the interior of X we get

$$
\begin{aligned}
\langle Tu - f, g \rangle &= \lim \langle Au_\nu - Au_\nu', g \rangle \\
&= \lim \langle u_\nu - u_\nu', A'g \rangle \\
&= 0,
\end{aligned}
$$

whence $Tu = f$.

We will think of T as an unbounded operator from V^i to V^{i+1}, whose domain is $\mathcal{D}_T^i$. Since $\mathcal{D}_T^i$ contains $V^i \cap C^\infty(X, E^i)$ the operator T is densely defined and closed.

From the lemma of Du Bois-Reymond and the uniqueness of a weak limit it follows that if $u \in \mathcal{D}_T^i$ then $Tu = Au$ in the sense of distributions in the interior of X.

Lemma 2.8 $T\mathcal{D}_T^i \subset \mathcal{D}_T^{i+1}$ and $T^2 = 0$.

Proof. Let $u \in \mathcal{D}_T^i$ and $\{u_\nu\}$ be a sequence with properties 1)–3). We set $f_\nu = Au_\nu$. Then $Tu = \lim f_\nu$. And since $Af_\nu = 0$, we obtain that $Tu \in \mathcal{D}_T^{i+1}$ and $T(Tu) = 0$.

$\square$

Thus we have the following complex of Hilbert spaces and their closed linear mappings:

$$
V^\cdot : \quad 0 \longrightarrow V^0 \xrightarrow{T} V^1 \xrightarrow{T} \ldots \xrightarrow{T} V^N \longrightarrow 0. \tag{2.5}
$$

The cohomology of the complex $\{E^i, A^i\}$ evaluated by the spaces $\{V^i\}$ is just the cohomology of the complex (2.5), that is

$$
H^i(V^\cdot) = \frac{\ker\{T : \mathcal{D}_T^i \to V^{i+1}\}}{T\mathcal{D}_T^{i-1}}.
$$

We now define T^*, the adjoint of T, as usual for unbounded operators. Namely, let $\mathcal{D}_{T^*}^i$ be the set of all $g \in V^i$ with the property that there is $v \in V^{i-1}$ satisfying $(Tu, g)_{V^i} = (u, v)_{V^{i-1}}$ for all $u \in \mathcal{D}_T^{i-1}$. We define $T^* : \mathcal{D}_{T^*}^i \to V^{i-1}$ by $T^*g = v$.

The operator T^* is well defined because the domain $\mathcal{D}_T^{i-1}$ is dense in V^{i-1}. It is clear that T^*g is in general different from A^*g in the sense of distributions in the interior of X, for A^* is formally adjoint for A in the sense of L^2-spaces on X.

Lemma 2.9 $T^* \mathcal{D}_{T^*}^i \subset \mathcal{D}_{T^*}^{i-1}$ *and* $T^{*2} = 0$.

Proof. Indeed, if $g \in \mathcal{D}_{T^*}^i$ and $u \in \mathcal{D}_T^{i-2}$ then by definition and Lemma 2.8 we get

$$
\begin{aligned}
(Tu, T^*g)_{V^{i-1}} &= (T(Tu), g)_{V^i} \\
&= 0.
\end{aligned}
$$

Therefore $T^*g \in \mathcal{D}_{T^*}^{i-1}$ and $T^*(T^*g) = 0$, which completes the proof.

$\square$

Thus we obtain the following (chain) complex of Hilbert spaces and their closed linear mappings:

$$
V^{\cdot *}: \ 0 \longleftarrow V^0 \overset{T^*}{\longleftarrow} V^1 \overset{T^*}{\longleftarrow} \ldots \overset{T^*}{\longleftarrow} V^N \longleftarrow 0. \tag{2.6}
$$

The complex (2.6) is called the adjoint complex for (2.5), and its homology is denoted by

$$
H_i(V^{\cdot *}) = \frac{\ker\{T^*: \mathcal{D}_{T^*}^i \to V^{i-1}\}}{T^* \mathcal{D}_{T^*}^{i+1}}.
$$

Let us introduce an operator L on V^i with a domain $\mathcal{D}_L^i$, which better suits the Hilbert structure of V^i than the formal Laplacian $\Delta = A^*A + AA^*$ of the complex $\{E^i, A^i\}$. Namely, write $\mathcal{D}_L^i$ for the set of all $u \in \mathcal{D}_T^i \cap \mathcal{D}_{T^*}^i$ with the property that $Tu \in \mathcal{D}_{T^*}^{i+1}$ and $T^*u \in \mathcal{D}_T^{i-1}$. Then the operator $L: \mathcal{D}_L^i \to V^i$ is defined by

$$
Lu = T^*Tu + TT^*u,
$$

cf. §4.2 in [Tar95a].

The Neumann problem for the complex $\{E^i, A^i\}$ in the spaces V^i consists in the following:

(NP) Given a section $f \in V^i$, when is there $u \in \mathcal{D}_L^i$ such that $Lu = f$, and how does u depend on f?

The weak orthogonal decomposition is actually the first step in solving the Neumann problem. Set

$$
\mathcal{H}^i = \{u \in \mathcal{D}_T^i \cap \mathcal{D}_{T^*}^i : \ Tu = T^*u = 0\},
$$

for $i = 0, 1, \ldots$. Since the operators T and T^* are closed, $\mathcal{H}^i$ is a closed subspace of V^i. Denote by $H: V^i \to \mathcal{H}^i$ the orthogonal projection of V^i onto $\mathcal{H}^i$.

Lemma 2.10 $u \in \mathcal{H}^i$ *if and only if* $u \in \mathcal{D}_L^i$ *and* $Lu = 0$.

Proof. If $u \in \mathcal{H}^i$ then obviously $u \in \mathcal{D}_L^i$ and $Lu = 0$. If $Lu = 0$ then $(Lu, u)_{V^i} = 0$, and since

$$(Lu, u)_{V^i} = \|Tu\|_{V^{i+1}}^2 + \|T^*u\|_{V^{i-1}}^2$$

we have $u \in \mathcal{H}^i$.

$\square$

Lemma 2.11 *The operator L is selfadjoint, and $(L + 1)^{-1}$ exists, is bounded, and is everywhere in V^i defined.*

Proof. Since T is a closed operator and the domain of T is dense, the same is also true for T^*, and $(T^*)^* = T$.

It follows that the operators $(TT^*+1)^{-1}$ and $(T^*T+1)^{-1}$ exist, are bounded, selfadjoint and defined everywhere in V^i, cf. [Yos65, p. 200].

We now easily verify that $(L+1)^{-1}$ exists, is bounded, is everywhere defined, and is given by the formula

$$(L + 1)^{-1} = (TT^* + 1)^{-1} + (T^*T + 1)^{-1} - 1,$$

cf. 4.2.4 in [Tar95a] and elsewhere.

$\square$

Corollary 2.12 (weak orthogonal decomposition) *The range of L is orthogonal to $\mathcal{H}^i$, and*

$$V^i = \mathcal{H}^i \oplus \overline{L\mathcal{D}_L^i}, \tag{2.7}$$

where $\overline{L\mathcal{D}_L^i}$ denotes the closure of $L\mathcal{D}_L^i$ in V^i.

Proof. This follows immediately from the selfadjointness of L and Lemma 2.10.

$\square$

In particular, if $L\mathcal{D}_L^i$ is closed then we get the "strong orthogonal decomposition"

$$V^i = \mathcal{H}^i \oplus T^*T\mathcal{D}_L^i \oplus TT^*\mathcal{D}_L^i. \tag{2.8}$$

Definition 2.13 *Let $L\mathcal{D}_L^i$ be closed and $f \in V^i$, then $f = Hf + Lu$ where $u \in \mathcal{D}_L^i$. The Neumann operator $N : V^i \to \mathcal{D}_L^i$ is defined by $Nf = u - Hu$.*

Note that N is well defined. Indeed, if also $f = Hf + Lu'$ where $u' \in \mathcal{D}_L^i$ then $L(u - u') = 0$ whence

$$
\begin{aligned}
(u - Hu) - (u' - Hu') &= (u - u') - H(u - u') \\
&= 0.
\end{aligned}
$$

We summarize the properties of the Neumann operator. They generalise those of the Green operator from Hodge theory, for the Neumann problem itself stems from the desire to extend the Hodge theory to the case of manifolds with boundary.

Lemma 2.14 *Suppose $L\mathcal{D}_L^i$ is closed. Then the Neumann operator N has the following properties:*

1) *N is bounded, selfadjoint, $HN = NH = 0$, and we have the orthogonal decomposition*

$$f = Hf + T^*TNf + TT^*Nf \qquad (2.9)$$

for all $f \in V^i$.

2) *If $f \in \mathcal{D}_T^i$ and $Tf = 0$ then $TNf = 0$. If moreover $L\mathcal{D}_L^{i+1}$ is closed then $TNf = NTf$.*

3) *If $f \in \mathcal{D}_{T^*}^i$ and $T^*f = 0$ then $T^*Nf = 0$. If moreover $L\mathcal{D}_L^{i-1}$ is closed then $T^*Nf = NT^*f$.*

Proof.

1) The equalities $HN = NH = 0$ and formula (2.9) follow immediately from the definition of N.

Further, by the Closed Graph Theorem there exists a constant $c > 0$ such that if $u \in \mathcal{D}_L^i$ is orthogonal to $\mathcal{H}^i$ then we have $\|Lu\| \geq c\|u\|$. Applying this to Nf, we obtain

$$
\begin{aligned}
\|Nf\| &\leq \frac{1}{c}\|LNf\| \\
&= \frac{1}{c}\|f - Hf\| \\
&\leq \frac{1}{c}\|f\|.
\end{aligned}
$$

Hence N is bounded.

Finally, the selfadjointness of N follows immediately from Lemma 2.11 because

$$
\begin{aligned}
(Nf, g)_{V^i} &= (Nf, Hg + LNg)_{V^i} \\
&= (Nf, LNg)_{V^i} \\
&= (LNf, Ng)_{V^i} \\
&= (f, Ng)_{V^i}.
\end{aligned}
$$

2) Let $f \in \mathcal{D}_L^i$. Then from (2.9) and Lemma 2.8 we get $T^*TNf \in \mathcal{D}_T^i$, and $Tf = 0$ implies $TT^*TNf = 0$. Hence it easily follows that $TNf = 0$.

If also $L\mathcal{D}_L^{i+1}$ is closed then for any $f \in \mathcal{D}_T^i$ we have $Tf = TT^*TNf$ on the one hand, and $Tf = TT^*NTf$ on the other hand. Hence it follows that $L(TNf - NTf) = 0$, and since $TNf - NTf$ is orthogonal to $\mathcal{H}^{i+1}$ we deduce that $TNf - NTf = 0$, as required.

3) The proof is analogous to that of part 2).

$\square$

If L^i is a hypoelliptic pseudodifferential operator in the interior of X then the harmonic space $\mathcal{H}^i$ consists of C^∞ sections in the interior of X, and the Neumann operator N preserves the interior regularity. Such is the case, in particular, if V^i are Sobolev spaces.

Beginning with its classical forms, the Dirichlet norm has been an important technical tool in studying the Neumann problem.

Given any $u, v \in \mathcal{D}_T^i \cap \mathcal{D}_{T^*}^i$, the Dirichlet inner product of u and v is defined by

$$D(u, v) = (Tu, Tv)_{V^{i+1}} + (T^*u, T^*v)_{V^{i-1}} + (u, v)_{V^i},$$

and the Dirichlet norm is $D(u) = \sqrt{D(u, u)}$.

The space $\mathcal{D}_T^i \cap \mathcal{D}_{T^*}^i$ with the Dirichlet norm is a complete (Hilbert) space. It is denoted by $\mathcal{D}^i$.

Since $D(u) \geq \|u\|_{V^i}$ for all $u \in \mathcal{D}^i$ there exists only one selfadjoint operator S with a domain $\mathcal{D}_S^i \subset \mathcal{D}^i$, such that if $u \in \mathcal{D}_S^i$ and $v \in \mathcal{D}^i$ then

$$D(u, v) = (Su, v)_{V^i}. \tag{2.10}$$

The following lemma gives a useful description of the operator L because our estimates will be in the norm $D(u)$.

Lemma 2.15 $\mathcal{D}_L^i = \mathcal{D}_S^i$ and $L = S - 1$, where the operator S is defined by (2.10).

Proof. If $u \in \mathcal{D}_L^i$ and $v \in \mathcal{D}^i$, then $D(u, v) = ((L+1)u, v)_{V^i}$ is fulfilled. Hence by the uniqueness of S, we have $S = L + 1$.
$\square$

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a vector space V. We will say that the norm $\|\cdot\|_1$ is completely continuous with respect to the norm $\|\cdot\|_2$ if every sequence which is bounded in the norm $\|\cdot\|_1$ has a convergent subsequence in the norm $\|\cdot\|_2$.

Lemma 2.16 If the norm D on $\mathcal{D}^i$ is completely continuous with respect to $\|\cdot\|_{V^i}$ then $\mathcal{H}^i$ is finite dimensional.

Proof. Observe that if $u, v \in \mathcal{H}^i$ then $D(u, v) = (u, v)_{V^i}$. Suppose that the dimension of $\mathcal{H}^i$ is infinite. Then there exists an infinite sequence $\{u_\nu\}$ of orthonormal elements in $\mathcal{H}^i$. Since $D(u_\nu) = \|u_\nu\|_{V^i} = 1$ the sequence $\{u_\nu\}$ contains a convergent subsequence. But this is at variance with the fact that if $\nu \neq \mu$ then $\|u_\nu - u_\mu\|_{V^i} = \sqrt{2}$.
$\square$

Lemma 2.17 If the norm D on $\mathcal{D}^i$ is completely continuous with respect to $\|\cdot\|_{V^i}$ then there exists a constant $c > 0$ such that for all $u \in D^i$ orthogonal to $\mathcal{H}^i$, we have

$$\|Tu\|_{V^{i+1}}^2 + \|T^*u\|_{V^{i-1}}^2 \geq c \|u\|_{V^i}^2.$$

Proof. Consider the Hilbert space $V^{i+1} \times V^{i-1}$ which is equipped with the norm

$$\|\{f, v\}\| = \left(\|f\|_{V^{i+1}}^2 + \|v\|_{V^{i-1}}^2 \right)^{1/2}.$$

Let $\mathfrak{M} \colon \mathcal{D}^i \to V^{i+1} \times V^{i-1}$ be the mapping defined by $\mathfrak{M}u = \{Tu, T^*u\}$. We note that $\mathfrak{M}$ is a closed operator.

We will prove that the range of $\mathfrak{M}$ is closed. Suppose that $\mathfrak{M}\mathcal{D}^i$ is not closed. Then there exists a sequence $\{u_\nu\}$ in $\mathcal{D}^i$, such that $\lim \mathfrak{M}u_\nu = \{f, v\}$ and $\{f, v\} \notin \mathfrak{M}\mathcal{D}^i$.

Set $u'_\nu = u_\nu - Hu_\nu$, then u'_ν are orthogonal to $\mathcal{H}^i$ and $\lim \mathfrak{M}u'_\nu = \{f, v\}$. If $\|u'_\nu\|_{V^i}$ are bounded then

$$D(u'_\nu) = (\|\mathfrak{M}u'_\nu\|^2 + \|u'_\nu\|_{V^i}^2)^{1/2}$$

are bounded, too. Then by hypothesis $\{u'_\nu\}$ has a convergent subsequence with a limit u, and since $\mathfrak{M}$ is closed then $\mathfrak{M}u = \{f, v\}$ which contradicts the assumption that $\{f, v\} \notin \mathfrak{M}\mathcal{D}^i$. Thus by choosing a subsequence, if necessary, we may assume that $\lim \|u'_\nu\|_{V^i} = \infty$.

Now set $U_\nu = u'_\nu / \|u'_\nu\|_{V^i}$. Then $\lim \|\mathfrak{M}U_\nu\| = 0$ and $D(U_\nu)$ are bounded. Therefore $\{U_\nu\}$ has a convergent subsequence $\{U_{\nu_k}\}$ such that

$$\begin{aligned} \lim U_{\nu_k} &= U, \\ \lim \mathfrak{M}U_{\nu_k} &= \{0, 0\}. \end{aligned}$$

Hence $\mathfrak{M}U = 0$ so that $U \in \mathcal{H}^i$. Since U_ν is orthogonal to $\mathcal{H}^i$ we have $U = 0$, but $\|U_\nu\|_{V^i} = 1$. This contradiction proves that the range $\mathfrak{M}\mathcal{D}^i$ is closed in $V^{i+1} \times V^{i-1}$.

Let R be the restriction of $\mathfrak{M}$ to the orthogonal complement of $\mathcal{H}^i$ in $\mathcal{D}^i$. Then R is one-to-one and has a closed range. By the Closed Graph Theorem, the inverse R^{-1} is bounded. Hence there is $c > 0$ such that $\|Ru\|^2 \geq c \|u\|_{V^i}^2$. This proves the lemma.

$\square$

Theorem 2.18 *If the norm D on $\mathcal{D}^i$ is completely continuous with respect to the norm $\|\cdot\|_{V^i}$, then $L\mathcal{D}^i$ is closed.*

Proof. By Lemma 2.17 there exists $c > 0$ with the property that for all $u \in \mathcal{D}_L^i$ which are orthogonal to $\mathcal{H}^i$ we have

$$(Lu, u)_{V^i} \geq c \|u\|_{V^i}^2,$$

so that $\|Lu\|_{V^i} \geq c \|u\|_{V^i}$.

Set $f = \lim Lu_\nu$. We may assume that u_ν are orthogonal to $\mathcal{H}^i$, and then $\|u_\nu\|_{V^i}$ are uniformly bounded. Therefore, $\{u_\nu\}$ has a subsequence whose

arithmetic means converge [1]. Denoting this limit by u, we get $f = Lu$, which completes the proof.

$\square$

The question of when the norm D on $\mathcal{D}^i$ is completely continuous with respect to the norm $\| \cdot \|_{V^i}$, is very difficult in the general case and it requires special consideration. We present some consequences here.

Corollary 2.19 *Suppose the norm D on $\mathcal{D}^i$ is completely continuous with respect to the norm $\|\cdot\|_{V^i}$. Then for the complex $\{E^i, A^i\}$ the Neumann problem is solvable at step i in the sense that there exist operators H and N in V^i with properties 1)–3) of Lemma 2.14.*

Proof. This follows immediately from Lemma 2.14 and Theorem 2.18.

$\square$

For compact manifolds with boundary X the subspace $\mathcal{H}^0$ is usually infinite-dimensional. So by Lemma 2.16 the Dirichlet norm D may not be completely continuous with respect to the norm $\| \cdot \|_{V^0}$ on $\mathcal{D}^0$. But the following result holds.

Theorem 2.20 *If the norm D on $\mathcal{D}^1$ is completely continuous with respect to the norm $\| \cdot \|_{V^1}$ then $L\mathcal{D}_L^0$ is closed.*

Proof. It suffices to prove that there exists a constant $c > 0$ such that $\|Lf\|_{V^0} \geq c \, \|f\|_{V^0}$ for all $f \in \mathcal{D}_L^0$ which are orthogonal to $\mathcal{H}^0$.

First, if $u \in \mathcal{D}_L^0$ then $Tu \in \mathcal{D}^1$ and $Tu \perp \mathcal{H}^1$. Thus by Lemma 2.17 we obtain

$$\begin{aligned} \|T^*Tu\|_{V^0}^2 &= \|Lu\|_{V^0}^2 \\ &\geq c \, \|Tu\|_{V^1}^2. \end{aligned}$$

Further, since $f \perp \mathcal{H}^0$ then by the weak orthogonal decomposition (2.7) $f \in \overline{L\mathcal{D}_L^0}$. Hence, for each $\varepsilon > 0$ there exists $u \in \mathcal{D}_L^0$ such that $\|f - Lu\|_{V^0} < \varepsilon$. Thus,

$$\begin{aligned} \|f\|_{V^0}^2 &\leq (Lu, f)_{V^0} + \varepsilon \, \|f\|_{V^0} \\ &\leq \|Tu\|_{V^1} \|Tf\|_{V^1} + \varepsilon \, \|f\|_{V^0} \\ &\leq \frac{1}{c} \, \|Lu\|_{V^0} \|Lf\|_{V^0} + \varepsilon \, \|f\|_{V^0} \\ &\leq \frac{1}{c} \, \|f\|_{V^0} \|Lf\|_{V^0} + \varepsilon \left(\frac{1}{c} \, \|Lf\|_{V^0} + \|f\|_{V^0} \right). \end{aligned}$$

---

[1]This actually puts some restrictions on the spaces V^i under study.

Since ε can be made arbitrarily small by choosing Lu close enough to f we obtain $\|Lf\|_{V^0} \geq c \,\|f\|_{V^0}$, which concludes the proof.

$\square$

The next result follows from Lemma 2.14 and Theorem 2.18. Recall that $\mathcal{H}^0 = \ker T^0$.

Corollary 2.21 *Suppose the norm D on $\mathcal{D}^1$ is completely continuous with respect to the norm $\| \cdot \|_{V^1}$. Then $f = Hf + T^*NTf$ for any section $f \in \mathcal{D}_T^0$, where $H : V^0 \to \mathcal{H}^0$ is the orthogonal projection.*

By assumption, the differential operator $A^0 = A$ has injective symbol. It follows that A^0 is hypoelliptic in the interior of X whence

$$\mathcal{H}^0 = \{u \in V^0 \cap C_{\mathrm{loc}}^\infty(\mathring{X}, E^0) : \ Au = 0\},$$

i.e., the operator H^0 is a generalisation of the Bergman projector from complex analysis. Corollary 2.21 gives

$$\begin{aligned} \pi \ &:= \ H^0 \\ &= \ I - T^*NT. \end{aligned}$$

As mentioned, a priori estimates for solutions of elliptic equations imply that for each interior point x of X the "evaluation functional" $\delta_x(u) = u(x)$ is bounded on $\mathcal{H}^0$. Therefore, $\mathcal{H}^0$ is a Hilbert space with reproducing kernel, cf. [Aro50].

Let $\{e_\nu\}_{\nu=1,2,\ldots}$ be some complete orthonormal system in $\mathcal{H}^0$. If $u \in \mathcal{H}^0$ then this section decomposes into the Fourier series $u = \sum c_\nu e_\nu$ which converges in the norm of the space V^0, and hence uniformly along with all derivatives on compact subsets of the interior of X. In the interior of $X \times X$ we consider the series

$$\begin{aligned} K(x,y) \ &= \ K_H(x,y) \\ &= \ \sum_{\nu=0}^\infty *_E \, e_\nu(x) \otimes e_\nu(y). \end{aligned} \tag{2.11}$$

Theorem 2.22 *Series (2.11) converges uniformly along with all derivatives on compact subsets of the interior of $X \times X$, so that*

$$K_H \in C_{\mathrm{loc}}^\infty(\mathring{X} \times \mathring{X}, E^* \boxtimes E).$$

If $x \in \mathring{X}$ is fixed, then this series actually converges in the norm of the space $E_x^* \otimes V^0$.

Proof. To shorten notation, we will restrict the discussion to the case where X is a closed domain in $\mathbb{R}^n$.

Let

$$e_\nu(x) = \begin{pmatrix} e_{\nu,1}(x) \\ \ldots \\ e_{\nu,k}(x) \end{pmatrix}$$

be representations of the sections e_ν, where k is the rank of E. Pick compact sets K_1 and K_2 in the interior of X. If $x \in K_1$ is a fixed point, then in view of the orthonormality of the system $\{e_\nu\}$ we obtain for $j = 1, \ldots, k$

$$
\begin{aligned}
\Big(\sum_{\nu=0}^{N} |e_{\nu,j}(x)|^2 \Big)^2 &\leq \Big| \sum_{\nu=0}^{N} \overline{e_{\nu,j}(x)} e_\nu(x) \Big|^2 \\
&\leq c_1 \Big\| \sum_{\nu=0}^{N} \overline{e_{\nu,j}(x)} e_\nu(y) \Big\|_{V^0}^2 \\
&= c_1 \sum_{\nu=0}^{N} |e_{\nu,j}(x)|^2,
\end{aligned}
$$

the constant $c_1 > 0$ depending on A and K only. Hence

$$\sum_{\nu=0}^{N} |e_{\nu,j}(x)|^2 \leq c_1 \tag{2.12}$$

for all $x \in K_1$.

Therefore, denoting by c_2 the constant obtained by analogy for the set K_2, we get for $(x, y) \in K_1 \times K_2$

$$
\begin{aligned}
\sum_{\nu=0}^{N} | *_E \ e_\nu(x) \otimes e_\nu(y)| &= \sum_{\nu=0}^{N} |e_\nu(x)| \, |e_\nu(y)| \\
&\leq k \sqrt{c_1 c_2}.
\end{aligned}
$$

This proves the absolute and uniform convergence of the series (2.11) on compact subsets of the interior of $X \times X$.

Finally, (2.12) implies that for fixed $x \in \overset{\circ}{X}$, the equality (2.11) gives the expansion of $K(x, y)$ in the complete orthonormal system $\{e_\nu\}$. To finish the proof, it is sufficient to observe that x and y enter into $K(x, y)$ in a symmetric way.

$\square$

Theorem 2.23 (Bergman formula) *If* $u \in \mathcal{H}^0$ *then*

$$u(x) = (u, K(x, \cdot))_{V^0}$$

for all $x \in \overset{\circ}{X}$.

Proof. Let $u = \sum c_\mu e_\mu$. Then by the previous theorem we get for fixed x in the interior of X

$$
\begin{aligned}
(u, K(x, \cdot))_{V^0} &= \sum_{\mu,\nu} c_\mu \, (e_\mu, e_\nu)_{V^0} \, e_\nu(x) \\
&= \sum_\nu c_\nu \, (e_\nu, e_\nu)_{V^0} \, e_\nu(x) \\
&= u(x),
\end{aligned}
$$

and the proof is complete. $\square$

Thus, in order to discover the properties of $\pi = H$ we might study the Neumann operator N^1. However, "good" properties of N^1 is not what we can generally expect. It is rather an instrument to produce examples for the general scheme.

2.4   Hodge theory on manifolds with boundary

Given a vector space V with norm $\| \cdot \|$, we write $C(V, \| \cdot \|)$ for the completion of V under the norm $\| \cdot \|$.

In this section we describe the Hodge theory of the Dirichlet problem for the Laplacian $\Delta = A^*A$ in the class of generalised sections of E on X. In order to define what is meant by the "solution" of the boundary value problem, we employ negative norms. This idea is certainly not new and goes back at least as far as [Sch60] and [Roi96].

Pick a Dirichlet system B_j, $j = 0, 1, \ldots, m-1$, of order $m-1$ on the boundary of X. More precisely, B_j is a differential operator of type $E \to F_j$ and order $m_j \le m-1$ in a neighbourhood U of ∂X. Moreover, the orders m_j are pairwise different and the symbols $\sigma(B_j)$, if restricted to the conormal bundle of ∂X, have ranks equal to the dimensions of F_j.

We actually assume that the dimensions of F_j are the same and equal to that of E.

Let C_j, $j = 0, 1, \ldots, m-1$, be the adjoint system for $\{B_j\}$ with respect to Green's formula, cf. [Tar95b]. Thus, C_j is a differential operator of type $F^* \to F_j^*$ and order $m - m_j - 1$ in a smaller neighbourhood U of ∂X. We now set

$$
\begin{aligned}
t(u) &= \bigoplus_{j=0}^{m-1} B_j u, \\
n(f) &= \bigoplus_{j=0}^{m-1} *_{F_j}^{-1} C_j *_F
\end{aligned}
$$

for $u \in C_{\text{loc}}^{m-1}(U, E)$ and $f \in C_{\text{loc}}^{m-1}(U, F)$.

Lemma 2.24 (Green's formula) *For each $u, v \in H^{2m}(X, E)$, the following formula holds:*

$$\int_{\partial X} \Big((t(u), n(Av))_x - (n(Au), t(v))_x \Big) ds = \int_X \Big((\Delta u, v)_x - (u, \Delta v)_x \Big) dx.$$

Proof. Cf. Corollary 9.2.12 in [Tar95b].

$\square$

Given F, we consider the boundary value problem

$$\begin{cases} \Delta u & = & F & \text{in} & X, \\ t(u) & = & 0 & \text{on} & \partial X, \end{cases} \tag{2.13}$$

which is an obvious generalisation of the classical Dirichlet problem, cf. 9.2.4 in [Tar95b].

Suppose $s > 0$. For sections $u \in C^\infty(X, E)$ we define two types of negative norms

$$\begin{aligned}
\|u\|_{-s} & = & \sup_{v \in C^\infty(X, E)} \frac{|(u, v)|}{\|v\|_s}, \\
|u|_{-s} & = & \sup_{\substack{v \in C^\infty(X, E) \\ t(v) = 0}} \frac{|(u, v)|}{\|v\|_s},
\end{aligned}$$

where $(\cdot, \cdot)$ is the scalar product in $L^2(X, E)$. We denote the completions of $C^\infty(X, E)$ with respect to these norms by $H^{-s}(X, E)$ and $C(C^\infty(X, E), |\cdot|_{-s})$, respectively. They are obviously Banach spaces and satisfy

$$H^{-s}(X, E) \hookrightarrow C(C^\infty(X, E), |\cdot|_{-s}),$$

for $\|u\|_{-s} \geq |u|_{-s}$.

We can define (u, v) for $u \in H^{-s}(X, E)$ and $v \in C^\infty(X, E)$ as follows. By definition, there is a sequence $\{u_\nu\}$ in $C^\infty(X, E)$ such that $\|u_\nu - u\|_{-s} \to 0$ as $\nu \to \infty$. Then

$$\begin{aligned}
|(u_\nu - u_\mu, v)| & \leq & \|u_\nu - u_\mu\|_{-s} \|v\|_s \\
& \to & 0
\end{aligned}$$

as $\mu, \nu \to \infty$. Set $(u, v) = \lim(u_\nu, v)$. Clearly, this limit does not depend on the particular sequence $\{u_\nu\}$, for if $\|u_\nu\|_{-s} \to 0$ then $|(u_\nu, v)| \leq \|u_\nu\|_{-s} \|v\|_s$ tends to zero, too. From the definition it follows that for all $u \in H^{-s}(X, E)$ and $v \in C^\infty(X, E)$ we get

$$|(u, v)| \leq \|u\|_{-s} \|v\|_s. \tag{2.14}$$

In a similar way we can define the pairing (u, v) for $u \in C(C^\infty(X, E), |\cdot|_{-s})$ and $v \in C^\infty(X, E)$ with $t(v) = 0$. Corresponding to (2.14) we have obviously $|(u, v)| \leq |u|_{-s} \|v\|_s$.

Let F be in $C(C^\infty(X, E), |\cdot|_{-s-2m})$, where $s \geq 0$. We shall say that $u \in H^{-s}(X, E)$ is a *strong solution* of (2.13) if there is a sequence of sections $u_\nu \in C^\infty(X, E)$ with $t(u_\nu) = 0$, such that

$$\begin{aligned} \|u_\nu - u\|_{-s} &\rightarrow 0, \\ |\Delta u_\nu - F|_{-s-2m} &\rightarrow 0 \end{aligned} \qquad (2.15)$$

as $\nu \to \infty$.

Denote by $\mathcal{H}(X)$ the set of all $u \in C^\infty(X, E)$ that satisfy $\Delta u = 0$ in the interior of X and $t(u) = 0$ on ∂X. Since (2.13) is an elliptic boundary value problem, $\mathcal{H}(X)$ is finite dimensional. Moreover, for any $u \in \mathcal{H}(X)$ we actually obtain

$$\begin{aligned} 0 &= (\Delta u, u) \\ &= (Au, Au) \end{aligned}$$

whence $Au = 0$ in X. Therefore, the space $\mathcal{H}(X)$ consists of all $u \in \mathcal{S}_A(\overset{\circ}{X})$ which are C^∞ up to the boundary of X and which vanish up to the infinite order on ∂X.

Lemma 2.25 *Let $s \geq 0$. If $F \in C(C^\infty(X, E), |\cdot|_{-s-2m})$ and $F \perp \mathcal{H}(X)$, then there is a strong solution $u \in H^{-s}(X, E)$ of (2.13) satisfying $u \perp \mathcal{H}(X)$ and*

$$\|u\|_{-s} \leq c \, |F|_{-s-2m}, \qquad (2.16)$$

where the constant c does not depend on F and u.

Proof. Cf. [Sch60].

$\square$

The definition (2.15) of a strong solution to (2.13) obviously corresponds to an appropriate closure $L : \mathcal{D}_L \to C(C^\infty(X, E), |\cdot|_{-s-2m})$ of the Laplacian $\Delta = A^* A$, cf. Chapter 2 in [Dez80]. Namely, we denote by $\mathcal{D}_L$ the set of all sections $u \in H^{-s}(X, E)$, for which there is a sequence $\{u_\nu\}$ with the following properties:

1) $u_\nu \in C^\infty(X, E)$ satisfies $t(u_\nu) = 0$;

2) $\{u_\nu\}$ converges to u in $H^{-s}(X, E)$; and

3) $\{\Delta u_\nu\}$ is a Cauchy sequence in $C(C^\infty(X, E), |\cdot|_{-s-2m})$.

The closed densely defined operator $L : \mathcal{D}_L \rightarrow C(C^\infty(X, E), |\cdot|_{-s-2m})$ given by $Lu = \lim \Delta u_\nu$, where $\{u_\nu\}$ is any sequence with properties 1)–3), is called the *strong extension* of Δ under the boundary conditions $t(u) = 0$. It is clear that $u \in H^{-s}(X, E)$ is a strong solution to problem (2.13) if and only if $Lu = F$.

It is worth pointing out that the case $\partial X = \emptyset$ is formally permitted in the following theorem.

Theorem 2.26 *Suppose* $s \geq 0$. *There are bounded linear operators*

$$
\begin{aligned}
H : \ & C(C^\infty(X, E), |\cdot|_{-s-2m}) \ \rightarrow \ \mathcal{H}(X), \\
G : \ & C(C^\infty(X, E), |\cdot|_{-s-2m}) \ \rightarrow \ \mathcal{D}_L
\end{aligned}
$$

such that

1) *H has the kernel $K_H(x, y) = \sum_\nu h_\nu(x) \otimes *_E h_\nu(y)$ where $\{h_\nu\}$ is an orthonormal basis of $\mathcal{H}(X)$;*

2) *$AH = 0$ and $GH = HG = 0$;*

3)
$$
\begin{aligned}
GLu \ &= \ u - Hu \quad \textit{for all} \ \ u \in \mathcal{D}_L, \\
LGF \ &= \ F - HF \quad \textit{for all} \ \ F \in C(C^\infty(X, E), |\cdot|_{-s-2m}).
\end{aligned}
$$

Proof. This follows by the same method as in Theorem 3.3 of [SST00], with Lemma 3.2 therefrom replaced by Lemma 2.25.
$\square$

The operators H and G are actually independent of s since they are unique extensions by continuity of these operators on the dense subspace $C^\infty(X, E)$ of $C(C^\infty(X, E), |\cdot|_{-s-2m})$.

When restricted to $L^2(X, E)$, the operator G is selfadjoint. Indeed, given any $F, v \in L^2(X, E)$, we may invoke the elliptic regularity of the Dirichlet problem (2.13) to conclude that both GF and Gv belong to $H^{2m}(X, E)$ and satisfy the boundary condition $t(\cdot) = 0$. It follows that $LGF = \Delta GF$ and $LGv = \Delta Gv$ whence

$$
\begin{aligned}
(GF, v) \ &= \ (GF, Hv + LGv) \\
&= \ (GF, A^* A\, Gv) \\
&= \ (A^* A\, GF, Gv) \\
&= \ (F, Gv) \,,
\end{aligned}
$$

which is due to Theorem 2.26. Hence the Schwartz kernel of G is Hermitean, i.e., $K_G(x, y)^* = K_G(y, x)$ for all (x, y) away from the diagonal of $X \times X$.

Corollary 2.27 *If, in addition, $F \in H^{-s-2m}(X, E)$, then there is a sequence of sections $u_\nu \in C^\infty(X, E)$ with $t(u_\nu) = 0$, such that*

$$\begin{aligned} \|u_\nu - u\|_{-s} &\to 0, \\ \|\Delta u_\nu - F\|_{-s-2m} &\to 0 \end{aligned} \tag{2.17}$$

as $\nu \to \infty$.

From Lemma 2.24 we deduce that when u is smooth enough, it fulfills $t(u) = 0$ if and only if $(\Delta u, v) = (u, \Delta v)$ for all v satisfying $t(v) = 0$. This gives rise to the concept of a *weak extension* of Δ under the boundary conditions $t(u) = 0$, cf. Chapter 2 in [Dez80]. Given an $F \in C(C^\infty(X, E), |\cdot|_{-s-2m})$, a section u is said to be a *weak solution* of (2.13) if it is in $H^{-s'}(X, E)$ for some $s' \geq 0$ and

$$(u, \Delta v) = (F, v) \tag{2.18}$$

for all $v \in C^\infty(X, E)$ satisfying $t(v) = 0$.

Lemma 2.28 *Suppose that $F \in C(C^\infty(X, E), |\cdot|_{-s-2m})$ where $s \geq 0$. If $u \in H^{-s'}(X, E)$ is a weak solution of (2.13), then actually $u \in H^{-s}(X, E)$ and it is a strong solution of (2.13). Moreover, there is a constant c not depending on F or u, such that*

$$\|u\|_{-s} \leq c \left(|F|_{-s-2m} + \|u\|_{-s'} \right).$$

Proof. Cf. [Sch60].

$\square$

To study the Dirichlet problem with nonzero boundary data $t(u) = u_0$ we need a result of [Roi96]. Denote by $H^{-s,B}(X, E)$ the completion of $C^\infty(X, E)$ with respect to the norm

$$\|u\|_{-s,B} := \|u\|_{-s} + \|t(u)\|_{\oplus H^{-s-m_j-\frac{1}{2}}(\partial X, F_j)} + \|n(Au)\|_{\oplus H^{-s-2m+m_j+\frac{1}{2}}(\partial X, F_j)}. \tag{2.19}$$

The advantage of using these spaces is that for each $u \in H^{-s,B}(X, E)$ there is a sequence $\{u_\nu\}$ in $C^\infty(X, E)$, such that $u_\nu \to u$ in $H^{-s}(X, E)$, and $\{t(u_\nu)\}$, $\{n(Au_\nu)\}$ are Cauchy sequences in $\oplus H^{-s-m_j-\frac{1}{2}}(\partial X, F_j)$ and $\oplus H^{-s-2m+m_j+\frac{1}{2}}(\partial X, F_j)$, respectively. Moreover, $\{\Delta u_\nu\}$ is a Cauchy sequence in $H^{-s-2m}(X, E)$, which follows by manipulations of Green's formula. Hence to any element $u \in H^{-s,B}(X, E)$ we can assign both $t(u)$, $n(Au)$ and Δu defined in the above strong sense.

Lemma 2.29 *For each pair*

$$u_0 \quad \in \quad \bigoplus_{j=0}^{m-1} H^{-s-m_j-\frac{1}{2}}(\partial X, F_j),$$

$$u_1 \quad \in \quad \bigoplus_{j=0}^{m-1} H^{-s-2m+m_j+\frac{1}{2}}(\partial X, F_j),$$

there is a section $u \in H^{-s,B}(X,E)$ with the property that $t(u) = u_0$ and $n(Au) = u_1$. Moreover, the mapping $(u_0, u_1) \mapsto U$ is continuous in the relevant norms.

Proof. See Lemma 6.1.2 in [Roi96].

$\square$

Given any

$$F \quad \in \quad C(C^\infty(X,E), |\cdot|_{-s-2m}),$$
$$u_0 \quad \in \quad \oplus H^{-s-m_j-\frac{1}{2}}(\partial X, F_j),$$

we now consider the inhomogeneous Dirichlet problem

$$\begin{cases} \Delta u & = & F & \text{in} & X, \\ t(u) & = & u_0 & \text{on} & \partial X. \end{cases} \tag{2.20}$$

A section u is said to be a *weak solution* of (2.20) if it is in $H^{-s'}(X,E)$ for some $s' \geq 0$ and

$$(u, \Delta v) = (F, v) - \int_{\partial X} (u_0, n(Av))_x ds \tag{2.21}$$

for all $v \in C^\infty(X,E)$ satisfying $t(v) = 0$.

Theorem 2.30 *Suppose $s \geq 0$. If $F \perp \mathcal{H}(X)$, then there is a weak solution $u \in H^{-s'}(X,E)$ to (2.20) with $u \perp \mathcal{H}(X)$. Moreover, $u \in H^{-s}(X,E)$ satisfies (2.20) in a strong sense, and there is a constant c independent of F, u_0 and u, such that*

$$\|u\|_{-s} \leq c \left(|F|_{-s-2m} + \|u_0\|_{\oplus H^{-s-m_j-\frac{1}{2}}(\partial X, F_j)} \right). \tag{2.22}$$

Proof. Using Lemma 2.29 we reduce (2.21) to (2.18) with a suitable right side F. To this end we choose $U \in H^{-s,B}(X,E)$ such that $t(U) = u_0$ and $n(AU) = u_1$, u_1 being arbitrary. By the above, there is a sequence $\{u_\nu\}$ in $C^\infty(X,E)$ such that

$$\begin{aligned} u_\nu & \rightarrow & U & \text{in} & H^{-s}(X,E), \\ t(u_\nu) & \rightarrow & u_0 & \text{in} & \oplus H^{-s-m_j-\frac{1}{2}}(\partial X, F_j), \\ n(Au_\nu) & \rightarrow & u_1 & \text{in} & \oplus H^{-s-2m+m_j+\frac{1}{2}}(\partial X, F_j) \end{aligned}$$

and $\Delta u_\nu \to F'$ in $H^{-s-2m}(X, E)$.

By Green's formula, we get

$$(u_\nu, \Delta v) = (\Delta u_\nu, v) - \int_{\partial X} (t(u_\nu), n(Av))_x \, ds$$

for all $v \in C^\infty(X, E)$ satisfying $t(v) = 0$. Letting $\nu \to \infty$ in this equality yields

$$(U, \Delta v) = (F', v) - \int_{\partial X} (u_0, n(Av))_x \, ds. \qquad (2.23)$$

Subtracting (2.23) from (2.21) we obtain

$$(u - U, \Delta v) = (F - F', v)$$

for all $v \in C^\infty(X, E)$ satisfying $t(v) = 0$, i.e., $u - U$ is a weak solution of the Dirichlet problem (2.13) with F replaced by $F - F'$. Moreover, it follows from (2.23) that

$$(F', v) = 0$$

for all $v \in \mathcal{H}(X)$. Combining Lemmas 2.28 and 2.25 thus results in the desired assertion. $\qquad\square$

We now derive a Poisson formula for solutions of the inhomogeneous Dirichlet problem.

To this end, we choose a Green operator $G_A(\cdot, \cdot)$ for A on X, cf. 9.2.1 in [Tar95b]. Given an oriented hypersurface $S \subset X$, we denote by $[S]^A$ the kernel on $X \times X$ defined by

$$\left\langle [S]^A, g \otimes u \right\rangle_{X \times X} = \int_S G_A(g, u)$$

for all $g \in C^\infty(X, F^*)$ and $u \in C^\infty(X, E)$ whose supports meet each other in a compact set.

In particular, the kernel $[\partial X]^A$ is obviously supported on the diagonal of $\partial X \times \partial X$.

For a section $u \in C^\infty(X, E)$, we set

$$\begin{aligned} (Mu)(x) &= -GA^* \left([\partial X]^A u \right) \\ &= -\int_{\partial X} G_A(K_{GA^*}(x, \cdot), u) \end{aligned}$$

when $x \in \overset{\circ}{X}$, K_{GA^*} being the Schwartz kernel of GA^*. The integral on the right-hand side is well defined, for K_{GA^*} is a C^∞ section of $E \boxtimes F^*$ outside of the diagonal of $X \times X$.

Corollary 2.31 *As defined above, M induces a continuous mapping P of $\oplus H^{-s-m_j-\frac{1}{2}}(\partial X, F_j)$ to $H^{-s}(X, E)$ such that $Pt(u) = Mu$. Moreover, for each weak solution u of (2.20) it follows that*

$$u = Hu + G\Delta u + Pt(u). \tag{2.24}$$

Proof. Let $u \in H^{-s'}(X, E)$ be a weak solution to (2.20). From Theorem 2.30 we deduce that $u \in H^{-s}(X, E)$ satisfies (2.20) in a strong sense. More precisely, there exists a sequence $u_\nu \in C^\infty(X, E)$ which approximates u in $H^{-s}(X, E)$, such that $t(u_\nu) \to t(u)$ and $\Delta u_\nu \to \Delta u$ in the relevant norms. We now set

$$
\begin{aligned}
Pu_0 &:= \lim_{\nu \to \infty} \left(u_\nu - Hu_\nu - G\left(\Delta u_\nu\right) \right) \\
&= u - Hu - G\left(\Delta u\right), \tag{2.25}
\end{aligned}
$$

the limit existing in $H^{-s}(X, E)$ by Theorem 2.26. Moreover, it is independent of the particular choice of u with a well-defined Δu and $t(u) = u_0$, which is again due to Theorem 2.26.

Obviously, $u_0 \mapsto Pu_0$ is a continuous mapping of $\oplus H^{-s-m_j-\frac{1}{2}}(\partial X, F_j)$ to $H^{-s}(X, E)$, and it remains to prove that it agrees with $-GA^*\left([\partial X]^A u\right)$ in the interior of X.

If $v \in C^\infty(X, E)$ has a compact support in the interior of X then by Theorem 2.26 we get

$$
\begin{aligned}
(Pu_0, v) &= (u, v) - (u, Hv) - (\Delta u, Gv) \\[2mm]
&= (u, v - Hv - \Delta(Gv)) - \int_{\partial X} (t(u), n(AGv))_x \, ds \\[2mm]
&= -\int_{\partial X} (t(u), n(AGv))_x \, ds,
\end{aligned}
$$

for $t(Gv) = 0$. The right-hand side here just amounts to $\left(-GA^*\left([\partial X]^A u\right), v\right)$, provided that u is smooth enough.

$\square$

From (2.25) it follows that Pu_0 is the unique solution to the Dirichlet problem

$$
\begin{cases}
\Delta u &= 0 \quad \text{in} \quad X, \\
t(u) &= u_0 \quad \text{on} \quad \partial X,
\end{cases}
$$

which is orthogonal to $\mathcal{H}(X)$. We call Pu_0 the Poisson integral of u_0. By Theorem 2.30,

$$\|Pu_0\|_{-s} \le c \, \|u_0\|_{\oplus H^{-s-m_j-\frac{1}{2}}(\partial X, F_j)} \tag{2.26}$$

with c a constant independent of u_0.

Theorem 2.32 *The quotient space $\mathcal{S}_{\Delta}^{(f)}(\overset{\circ}{X})/\mathcal{H}(X)$ is topologically isomorphic to $\oplus_{j=0}^{m-1} \mathcal{D}'(\partial X, F_j)$.*

Proof. Given any $u \in \mathcal{S}_{\Delta}^{(f)}(\overset{\circ}{X})$, the Cauchy data $t(u)$ and $n(Au)$, being first defined near ∂X, have weak limit values u_0 and u_1 on ∂X belonging to $\oplus_{j=0}^{m-1} \mathcal{D}'(\partial X, F_j)$, cf. [Tar95b, 9.4]. Pick a regularisation of u on ∂X, i.e., any section $U \in H^{-s'}(X, E)$ which coincides with u in the interior of X, cf. [Tar95b, 9.3.6].

Using the parametrix G of Δ given by Theorem 2.26, we get by Green's formula

$$u(x) - HU(x) = -\int_{\partial X} \Big((u_0, n(AK_G(\cdot, x)))_y - (u_1, t(K_G(\cdot, x)))_y \Big) ds$$

for $x \in \overset{\circ}{X}$. Since $t(K_G(\cdot, x)) = 0$ for all $x \in \overset{\circ}{X}$, it follows that $u = HU + Pu_0$, the section $HU \in \mathcal{H}(X)$ being independent of the particular choice of the regularisation U.

We have thus proved that any solution $u \in \mathcal{S}_{\Delta}^{(f)}(\overset{\circ}{X})$ is representable through the weak limit values $t(u)$ on ∂X by the Poisson formula (2.24). Furthermore, $\oplus_{j=0}^{m-1} \mathcal{D}'(\partial X, F_j)$ is the inductive limit of the sequence $\oplus_{j=0}^{m-1} H^{-s-m_j-\frac{1}{2}}(\partial X, F_j)$, $s \in \mathbb{N}$, for the boundary of X is a compact closed C^{∞} manifold. Combining this with (2.26) we deduce that the mapping $u \mapsto t(u)$ gives the desired isomorphism.

$\square$

Since $\mathcal{S}_A^{(f)}(\overset{\circ}{X})$ is obviously a closed subspace of $\mathcal{S}_{\Delta}^{(f)}(\overset{\circ}{X})$, the mapping

$$\frac{\mathcal{S}_A^{(f)}(\overset{\circ}{X})}{\mathcal{H}(X)} \to \bigoplus_{j=0}^{m-1} \mathcal{D}'(\partial X, F_j)$$

given by $u \mapsto t(u)$ identifies the quotient space with a closed subspace of $\oplus_{j=0}^{m-1} \mathcal{D}'(\partial X, F_j)$.

2.5 Hardy spaces

Suppose $U \subset\subset \overset{\circ}{X}$ is a domain with C^{∞} boundary. Fix a Dirichlet system $B = \{B_j\}_{j=0}^{m-1}$ of order $m-1$ on ∂U, each B_j being a differential operator of order m_j and type $E \to F_j$ in a neighbourhood $\mathcal{N}$ of ∂U. For a section u of E near ∂U we set

$$t(u) = \bigoplus_{j=0}^{m-1} B_j u \mid_{\partial U},$$

if defined.

Since $\Delta = A^*A$ satisfies the condition $(U)_s$ in the interior of X, the sesquilinear form

$$h(u, v) = \int_{\partial U} (t(u), t(v))_x ds \qquad (2.27)$$

defines a scalar product on $\mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E)$. Denote by $H_\Delta^{(B)}(U)$ the completion of $\mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E)$ in the norm $u \mapsto \sqrt{h(u, u)}$. These spaces are called the *Hardy spaces*, by analogy to the classical Hardy spaces of harmonic functions. Alternatively $H_\Delta^{(B)}(U)$ can be described as the space of all $u \in \mathcal{S}_\Delta(U)$ of finite order of growth, for which the weak boundary values of $t(u)$ belong to $\oplus_{j=0}^{m-1} L^2(\partial U, F_j)$.

Lemma 2.33 $H_\Delta^{(B)}(U)$ *is a separable Hilbert space with a reproducing kernel.*

Proof. By the very definition, $H_\Delta^{(B)}(U)$ can be identified as a closed subspace in $\oplus L^2(\partial U, F_j)$. In particular, $H_\Delta^{(B)}(U)$ is a separable Hilbert space because $\oplus L^2(\partial U, F_j)$ is.

Theorem 2.32 implies that each element $u_0 \in H_\Delta^{(B)}(U)$ can be actually thought of as a solution from $\mathcal{S}_\Delta^{(f)}(U)$. To make this more precise we invoke Theorem 2.30, with U in place of X. Since $\mathcal{H}(U)$ is trivial in this case, and $L^2(\partial U, F_j) \hookrightarrow H^{-m_j}(\partial U, F_j)$ for $j = 0, 1, \ldots, m-1$, there is a unique section $u \in H^{1/2}(U, E)$ satisfying $\Delta u = 0$ and $t(u) = u_0$ in a strong sense. Moreover, we have

$$\|u\|_{H^{1/2}(U,E)} \le c \, \|u_0\|_{\oplus H^{-m_j}(\partial U, F_j)}$$

with c a constant independent of u_0.

By (2.24), we get $u = P_U u_0$ where P_U is the Poisson integral related to the domain U. This gives us the desired identification of $H_\Delta^{(B)}(U)$ within $\mathcal{S}_\Delta^{(f)}(U)$, for P_U is a topological isomorphism of $\oplus_{j=0}^{m-1} \mathcal{D}'(\partial U, F_j)$ onto $\mathcal{S}_\Delta^{(f)}(U)$, cf. Theorem 2.32.

Using this fact we easily conclude that for any $x \in U$ all the evaluation functionals $u \mapsto \delta_x^{(j)} u := u_j(x)$, $j = 1, \ldots, k$, are continuous on $H_\Delta^{(B)}(U)$. Moreover, a stronger property than the continuity holds. Namely, for each compact set $K \subset U$ there is a constant C_K such that $\|\delta_x^{(j)}\| < C_K$ for all $x \in K$. Hence, $H_\Delta^{(B)}(U)$ is a Hilbert space with a reproducing kernel, cf. [Aro50]. $\square$

Obviously, $\mathcal{S}_A(U) \subset \mathcal{S}_\Delta(U)$ holds. Denote by $H_A^{(B)}(U)$ the Hardy space for A in U, i.e., the subspace of $H_\Delta^{(B)}(U)$ consisting of all solutions to $Au = 0$ in U.

Lemma 2.34 $H_A^{(B)}(U)$ *is a separable Hilbert space with a reproducing kernel.*

Proof. The statement will follow from Lemma 2.33 once we prove that $H_A^{(B)}(U)$ is a closed subspace in $H_\Delta^{(B)}(U)$. This latter assertion is a consequence of the Poisson formula (2.24) which shows that the original topology of $H_\Delta^{(B)}(U)$ is finer than that induced from $C_{\mathrm{loc}}^\infty(U, E)$.

$\square$

As mentioned, the Dirichlet problem for Δ in U is uniquely solvable, and its Hodge parametrix G_U is, in fact, a Green function of U, i.e., $H_U = 0$.

Lemma 2.35 *Let $K(\cdot, \cdot)$ be the reproducing kernel of $H_\Delta^{(B)}(U)$. Then for all $(x, y) \in U \times U$ we have*

$$K(x, y) = \int_{\partial U} \left(n(AK_{G_U}(\cdot, x)), n(AK_{G_U}(\cdot, y)) \right)_z ds. \qquad (2.28)$$

Moreover, for every fixed $x \in U$, the columns of the matrix $K(x, \cdot)$ belong to $\mathcal{S}_\Delta(U) \cap C_{\mathrm{loc}}^\infty(\overline{U}, E)$.

Proof. Note that the integral on the right-hand side of (2.28) is well defined for all $(x, y) \in U \times U$, and it belongs to $E_x^* \otimes \mathcal{S}_\Delta(U)$ in y, for every fixed $x \in U$. By the elliptic regularity, it is actually in $E_x^* \otimes C_{\mathrm{loc}}^\infty(\overline{U}, E)$, as claimed.

Let $u \in H_\Delta^{(B)}(U)$. Combining the Poisson formula (2.24) and the Bergman formula of Theorem 2.23, we get

$$0 = \int_{\partial U} \left((t(u), n(AK_{G_U}(\cdot, x)))_z + (t(u), t(K(x, \cdot)))_z \right) ds$$

for all $x \in U$. Hence it follows that $t(K(x, \cdot)) = -n(AK_{G_U}(\cdot, x))$ on ∂U, for each $x \in U$. We thus have

$$\begin{cases} \Delta K(x, \cdot) &= 0 & \text{in} \quad U, \\ t(K(x, \cdot)) &= -n(AK_{G_U}(\cdot, x)) & \text{on} \quad \partial U \end{cases} \qquad (2.29)$$

for every fixed $x \in U$. Once again using the Poisson formula we arrive at (2.28), as desired.

$\square$

It is obvious that

$$\left(\int_{\partial U} \left(n(AK_{G_U}(\cdot, x)), n(AK_{G_U}(\cdot, y)) \right)_z ds \right)^*$$
$$= \int_{\partial U} \left(n(AK_{G_U}(\cdot, y)), n(AK_{G_U}(\cdot, x)) \right)_z ds,$$

which recovers the equality $K(x, y)^* = K(y, x)$.

3   Duality for solutions of finite order of growth

In this section we describe the dual space of $\mathcal{S}_A^{(f)}(U)$ by using various pairings in Hilbert spaces of solutions to the system $Au = 0$ in U.

3.1   Pairing in Hardy spaces

According to the general scheme we will study in this and the next subsection the case

$$
\begin{aligned}
\Sigma_1 &= \mathcal{S}_\Delta^{(f)}(U), \\
\Sigma_2 &= \mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E).
\end{aligned}
$$

Let $V = H_\Delta^{(B)}(U)$. The Poisson formula implies that each element of $H_\Delta^{(B)}(U)$ belongs to $\mathcal{S}_\Delta^{(f)}(U)$. Write

$$
\begin{aligned}
i_1 &: & H_\Delta^{(B)}(U) &\to \mathcal{S}_\Delta^{(f)}(U), \\
i_2 &: \mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E) &\to H_\Delta^{(B)}(U)
\end{aligned}
$$

for the canonical embeddings.

The mappings i_1 and i_2 are always injective. As mentioned, the Poisson formula (2.24) implies the continuity of i_1. The mapping i_2 is continuous because the topology of $\oplus C^\infty(\partial U, F_j)$ is stronger than that of $\oplus L^2(\partial U, F_j)$. Moreover, it follows from Theorem 2.32 that $\mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E)$ is dense in $\mathcal{S}_\Delta^{(f)}(U)$.

The task is now to extend the sesquilinear pairing $h(\cdot, \cdot)$, cf. (2.27), from $H_\Delta^{(B)}(U) \times H_\Delta^{(B)}(U)$ to $\Sigma_1 \times \Sigma_2$.

Choose a smooth real-valued function $\varrho(x)$ on X with the property that $U = \{x \in X : \varrho(x) < 0\}$ and $\nabla \varrho(x) \neq 0$ on ∂U. Set

$$
U_\varepsilon = \{x \in X : \varrho(x) < \varepsilon\},
$$

then $U_{-\varepsilon} \Subset U \Subset U_\varepsilon$ for all sufficiently small $\varepsilon > 0$, and $\partial U_{\pm\varepsilon}$ is as smooth as ∂U.

Lemma 3.1 *Given any solutions $u \in \mathcal{S}_\Delta^{(f)}(U)$ and $v \in \mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E)$, the limit*

$$
\tilde{h}(u, v) = \lim_{\varepsilon \to 0+} \int_{\partial U_{-\varepsilon}} (t(u), t(v))_x \, ds_{-\varepsilon} \tag{3.1}
$$

exists. The corresponding pairing $\tilde{h} : \Sigma_1 \times \Sigma_2 \to \mathbb{C}$ is separately continuous, and its restriction to $H_\Delta^{(B)}(U) \times H_\Delta^{(B)}(U)$ coincides with h.

Proof. We first note that the limit (3.1) is none other than the definition of weak boundary values $t(u)$. Therefore, its existence and the separate continuity

of the corresponding pairing $\tilde{h}$ is a direct consequence of this definition. The second part of the lemma follows immediately from the definition of Hardy space $H_\Delta^{(B)}(U)$.

$\square$

Theorem 3.2 *The mapping* $\mathcal{J}: \ \mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E) \to \mathcal{S}_\Delta^{(f)}(U)'$ *induced by (3.1) is a topological isomorphism of these spaces.*

Proof. Since the natural inclusion i_2 is one-to-one, the mapping $\mathcal{J}$ is one-to-one, too, cf. Lemma 2.2.

In this concrete situation we can easily prove the surjectivity of the mapping $\mathcal{J}$ directly, i.e., without using Theorem 2.3.

Indeed, Theorem 2.32 states that the mapping $t: \ \mathcal{S}_\Delta^{(f)}(U) \to \oplus \mathcal{D}'(\partial U, F_j)$ is a topological isomorphism, with t^{-1} given by the Poisson integral P_U, cf. (2.24). Let $\mathcal{F}$ be a continuous linear functional on $\mathcal{S}_\Delta^{(f)}(U)$. Then the composition $\mathcal{F} \circ P_U$ is a continuous linear functional on $\oplus \mathcal{D}'(\partial U, F_j)$. Since $\oplus \mathcal{D}(\partial U, F_j^*)$ is a reflexive space, there is an element $v_0 \in \oplus \mathcal{D}(\partial U, F_j^*)$ such that

$$\langle \mathcal{F} \circ P_U, u_0 \rangle = \langle u_0, v_0 \rangle$$

for all $u_0 \in \oplus \mathcal{D}'(\partial U, F_j)$.

Set

$$v = P_U *_{\oplus F_j}^{-1} v_0,$$

then $v \in \mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E)$, which is due to the properties of the Poisson integral. If $u \in \mathcal{S}_\Delta^{(f)}(U)$ then

$$
\begin{aligned}
\mathcal{F}(u) &= \langle \mathcal{F}, P_U t(u) \rangle \\
&= \langle \mathcal{F} \circ P_U, t(u) \rangle \\
&= \langle t(u), v_0 \rangle \\
&= \int_{\partial U} (t(u), t(v))_x \, ds \\
&= \tilde{h}(u, v),
\end{aligned}
$$

i.e., $\mathcal{J}v = \mathcal{F}$. This proves that $\mathcal{J}$ is surjective.

We have thus proved that the mapping $\mathcal{J}$ is an isomorphism of the vector spaces $\mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E)$ and $\mathcal{S}_\Delta^{(f)}(U)'$. Moreover, both $\mathcal{J}$ and $\mathcal{J}^{-1}$ are continuous, which is clear from the explicit construction of $\mathcal{J}$. This completes the proof.

$\square$

Corollary 3.3 *The mapping* $\mathcal{J}' : \; \mathcal{S}_\Delta^{(f)}(U) \to \left(\mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E) \right)'$ *induced by (3.1) is a topological isomorphism of these spaces.*

Proof. According to Theorem 2.32 the space $\mathcal{S}_\Delta^{(f)}(U)$ is reflexive, i.e., under the natural pairing, we have

$$\left(\mathcal{S}_\Delta^{(f)}(U)' \right)' \overset{\text{top.}}{\cong} \mathcal{S}_\Delta^{(f)}(U),$$

where both $\mathcal{S}_\Delta^{(f)}(U)'$ and $(\mathcal{S}_\Delta^{(f)}(U)')'$ are endowed with the strong topology of the dual space. Thus, the desired statement follows immediately from Theorem 3.2.

$\square$

Since we have a commutative diagram

$$
\begin{array}{ccccc}
\mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E) & \overset{i_2}{\hookrightarrow} & H_\Delta^{(B)}(U) & \overset{i_1}{\hookrightarrow} & \mathcal{S}_\Delta^{(f)}(U) \\
\cup & & \cup & & \cup \\
\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E) & \overset{i_2}{\hookrightarrow} & H_A^{(B)}(U) & \overset{i_1}{\hookrightarrow} & \mathcal{S}_A^{(f)}(U)
\end{array}
$$

the pairing $\tilde{h}(\cdot, \cdot)$ induces a continuous mapping

$$\mathfrak{J} : \; \mathcal{S}_A(U) \cap C^\infty(\overline{U}, E) \to \mathcal{S}_A^{(f)}(U)'$$

which is the restriction of $\mathcal{J}$.

Write π for the Szegö projection

$$\pi : \; H_\Delta^{(B)}(U) \to H_A^{(B)}(U).$$

Corollary 3.4 *The mapping* $\mathfrak{J}$ *is a topological isomorphism of the space* $\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E)$ *onto* $\mathcal{S}_A^{(f)}(U)'$ *if and only if*

1) $\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E)$ *is dense in* $\mathcal{S}_A^{(f)}(U)$;
2) π *maps* $\mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E)$ *continuously into* $\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E)$.

Proof. Set

$$
\begin{aligned}
\mathfrak{S}_1 &= \mathcal{S}_A^{(f)}(U), \\
\mathfrak{S}_2 &= \mathcal{S}_A(U) \cap C^\infty(\overline{U}, E), \\
\mathfrak{V} &= H_A^{(B)}(U)
\end{aligned}
$$

and apply Corollary 2.7.

$\square$

Example 3.5 Let $X = \mathbb{R}^n$, $n \geq 3$, and let U be a ball. Consider a Dirac operator A in $\mathbb{R}^n$, i.e., a homogeneous first order differential operator with constant coefficients, such that $-A^*A$ is a diagonal matrix with the usual Laplace operator on the diagonal.

It is proved in [Shl96] that there are systems $\{b_\nu\}$ and $\{c_\mu\}$ of ($\mathbb{C}^k$-valued) homogeneous harmonic polynomials, such that

a) $\{b_\nu\}$ is an orthogonal basis in all the spaces $\mathcal{S}_A^{(s)}(U)$, $s \in \mathbb{Z}_+$, simultaneously, and an orthonormal basis in $H_A^{(B)}(U)$;

b) $\{b_\nu\} \cup \{c_\mu\}$ is an orthogonal basis in all the spaces $\mathcal{S}_\Delta^{(s)}(U)$, $s \in \mathbb{Z}_+$, simultaneously, and an orthonormal basis in $H_\Delta^{(B)}(U)$.

Property $a)$ implies, in particular, that condition 1) of Corollary 3.4 is fulfilled. Moreover, the projection π, if restricted to $\mathcal{S}_\Delta^{(s)}(U)$, coincides with the $H^s(U, E)$-orthogonal projection $\mathcal{S}_\Delta^{(s)}(U) \to \mathcal{S}_A^{(s)}(U)$. It follows that π maps $\mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E)$ continuously into $\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E)$. The isomorphism of Corollary 3.4 holds for A.

Example 3.6 Let A be a determined elliptic operator such that both A and A^* possess the Unique Continuation Property. Then condition 1) of Corollary 3.4 holds true.

Write $H_{A^*}^{(C)}(U)$ for the closed subspace of the Hardy space $H_{AA^*}^{(C)}(U)$ consisting of all solutions to $A^*g = 0$ in U. We consider the extension of A to an operator

$$T : \ H_\Delta^{(B)}(U) \to H_{A^*}^{(C)}(U),$$

whose domain $\mathcal{D}_T$ consists of all $u \in H_\Delta^{(B)}(U)$ with the property that there is a sequence $\{u_\nu\}$ in $\mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E)$, such that

1) $u_\nu \to u$ in $H_\Delta^{(B)}(U)$;
2) Au_ν is a Cauchy sequence in $H_{A^*}^{(C)}(U)$.

Using the existence of a two-sided fundamental solution Φ for A, one easily verifies that T is a densely defined closed operator.

Let P_U denote the Poisson integral of the Dirichlet problem for A^*A in U. Then the adjoint

$$T^* : \ \mathcal{D}_{T^*} \to H_\Delta^{(B)}(U)$$

is given by

$$T^*g = P_U \, n \, (AP_U \, n(g))$$

for every $g \in \mathcal{S}_{A^*}(U) \cap C^\infty(\overline{U}, F)$. Indeed, by the elliptic regularity of the Dirichlet problem we deduce that $P_U \, n \, (AP_U \, n(g)) \in \mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E)$ for each $g \in \mathcal{S}_{A^*}(U) \cap C^\infty(\overline{U}, F)$. Then

$$
\begin{aligned}
\int_{\partial U} (t(u), t(P_U \, n(AP_U \, n(g))))_x \, ds &= \int_{\partial U} G_A(*_F AP_U \, n(g), u) \\
&= \int_{\partial U} G_{A^*}(*_E P_U \, n(g), Au) \\
&= \int_{\partial U} (n(Au), t(P_U \, n(g)))_x \, ds \\
&= \int_{\partial U} (n(Au), n(g))_x \, ds
\end{aligned}
$$

for all $u \in \mathcal{D}_T$. In particular, this means that T^* is a closed densely defined operator.

Further, the existence of a fundamental solution implies that the range of T is closed and equals to $H_{A^*}^{(C)}(U)$. Hence T^* and $L^1 = TT^*$ are one-to-one operators with closed range.

It is clear that $TT^* = (AP_U\, n)^2$. To identify the operator $N^1 = (TT^*)^{-1}$, we denote by P_U^1 the Poisson integral of the Dirichlet problem for AA^* in U. Then we have

$$(AP_U\, n)^{-1}\, g = P_U^1\, t(\Phi_U g)$$

for all $g \in H_{A^*}^{(C)}(U)$. Indeed, by the properties of the Poisson integral we see that

$$
\begin{aligned}
n\left(P_U^1\, t(\Phi_U g)\right) &= t\left(\Phi_U g\right), \\
P_U\, t\left(\Phi_U g\right) &= \Phi_U g, \\
AP_U\, t\left(\Phi_U g\right) &= g
\end{aligned}
$$

because $A^* A \Phi_U g = A^* g = 0$ in U.

Finally, since Φ has the transmission property with respect to ∂U and the Dirichlet problem is elliptic we conclude that $(AP_U\, n)^{-1}$ maps $C^\infty(\overline{U}, F)$ to itself. Therefore, the projection π maps $\mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E)$ continuously into $\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E)$, cf. Section 2.3. By Corollary 3.4, $\mathfrak{J}$ is a topological isomorphism of

$$\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E) \overset{\cong}{\to} \mathcal{S}_A^{(f)}(U)'.$$

Example 3.7 Suppose A is included into an elliptic compatibility complex of differential operators $A^i \in \mathrm{Diff}^{m_i}(X; E^i, E^{i+1})$, $i = 0, 1, \ldots, N$, over X, with $A^0 = A$. As usual we introduce the Laplace operators $\Delta^i = A^{i*}A^i + A^{i-1}A^{i-1*}$ for every i. They are not elliptic in general, for the orders m_i may be pairwise different.

However, any Δ^i admit a well-posed Dirichlet problem, cf. Section 2.4. Namely, denote by B^i the boundary system consisting of the Cauchy data with respect to A^i and the Cauchy data with respect to A^{i-1*} on ∂U. In the notation of [Tar95a], these are $t(u)$ and $n(u)$, respectively. It is easy to verify that if $\Delta^i u = 0$ in U and $t(u) = 0$, $n(u) = 0$ on ∂U then u actually satisfies $A^i u = 0$ and $A^{i-1*} u = 0$ in U. Since the complex is elliptic we deduce that u is a C^∞ section of E^i with a support in $\overline{U}$.

Suppose any Laplacian Δ^i has the property $(U)_s$ in $\overset{\circ}{X}$. Then we can introduce Hardy spaces $H_{\Delta^i}^{(B^i)}(U)$ as in Section 2.5. Since $A\Delta = \Delta A$, the differential A preserves the elements of $H_{\Delta^i}^{(B^i)}(U)$ that are sufficiently smooth up to the boundary. Hence $\{E^i, A^i\}$ gives rise to a complex of closed operators in Hilbert spaces

$$0 \longrightarrow H_\Delta^{(B)}(U) \overset{T}{\longrightarrow} H_{\Delta^1}^{(B^1)}(U) \overset{T}{\longrightarrow} \ldots \overset{T}{\longrightarrow} H_{\Delta^N}^{(B^N)}(U) \longrightarrow 0, \qquad (3.2)$$

cf. Section 2.3.

The complex (3.3) has a distinguished subcomplex corresponding to the subspaces of $H_{\Delta^i}^{(B^i)}(U)$ with $t(u) = 0$. Factorising (3.3) by this subcomplex leads to the so-called *tangential complex* on ∂U. More precisely, the spaces of the tangential complex are those subspaces of $H_{\Delta^i}^{(B^i)}(U)$ which consist of sections with $n(u) = 0$, while its differential T_b is a quotient of T. By the unique solvability of the Dirichlet problem for Δ^i in U, these latter subspaces can be specified as spaces of L^2-sections of certain vector bundles E_b^i over the boundary of U. In particular, $E_b^0 = \oplus_{j=0}^{m-1} F_j \mid_{\partial U}$. In fact, the tangential complex corresponds to a complex $\{E_b^i, A_b^i\}$ of differential operators on ∂U, cf. [Tar95a, 3.1.5]. We get

$$0 \longrightarrow L^2(\partial U, E_b^0) \xrightarrow{T_b} L^2(\partial U, E_b^1) \xrightarrow{T_b} \ldots \xrightarrow{T_b} L^2(\partial U, E_b^{N-1}) \longrightarrow 0, \quad (3.3)$$

so that $H_A^{(B)}(U) \cong \{u \in L^2(\partial U, E_b^0) : T_b u = 0\}$.

If the domain $U \subset\subset X$ is strictly pseudoconvex relative to the Levi form of $\{E^i, A^i\}$ at step 1, then the Neumann problem for this complex in U is solvable at step 1. Moreover, the Neumann operator N preserves C^∞ sections of F up to the boundary, cf. [Tar95a, 4.1.5]. Hence the operator $I - T_b^* N T_b$, i.e., the orthogonal projection from $L^2(\partial U, E_b^0)$ to $\ker T_b^0$, maps $C^\infty(\overline{U}, E)$ continuously to itself. As $L^2(\partial U, E_b^0) = H_\Delta^{(B)}(U)$ we see that the projection π has the same property.

The question on the density of $\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E)$ in $\mathcal{S}_A^{(f)}(U)$ requires a separate discussion, cf. for instance [Tar95b]. We only mention that this is the case if A is a differential operator with constant coefficients in $\mathbb{R}^n$ and $U \subset\subset \mathbb{R}^n$ is convex.

Thus, we can invoke Corollary 3.4 to see that the mapping $\mathfrak{J}$ is a topological isomorphism of $\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E)$ onto $\mathcal{S}_A^{(f)}(U)'$. By reflexivity, the transpose $\mathfrak{J}'$ gives a topological isomorphism of the dual of $\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E)$ onto $\mathcal{S}_A^{(f)}(U)$.

3.2 Pairing in Lebesgue spaces

As before, we consider

$$\begin{aligned}
\Sigma_1 &= \mathcal{S}_\Delta^{(f)}(U), \\
\Sigma_2 &= \mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E).
\end{aligned}$$

Let $V = \mathcal{S}_\Delta^{(0)}(U)$. Again the Poisson formula shows that each element of $\mathcal{S}_\Delta^{(0)}(U)$ belongs to $\mathcal{S}_\Delta^{(f)}(U)$. Write

$$\begin{aligned}
i_1 &: & \mathcal{S}_\Delta^{(0)}(U) &\to \mathcal{S}_\Delta^{(f)}(U), \\
i_2 &: \mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E) &\to \mathcal{S}_\Delta^{(0)}(U)
\end{aligned}$$

for the natural inclusions.

The mappings i_1 and i_2 are always injective. As is already mentioned, the Poisson formula (2.24) implies the continuity of i_1. The mapping i_2 is continuous, too, because the topology of $C^\infty(\overline{U}, E)$ is finer than the topology of $L^2(U, E)$.

Moreover, it follows from Theorem 2.32 that $\mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E)$ is dense in $\mathcal{S}_\Delta^{(f)}(U)$.

Our next task is to extend the natural sesquilinear pairing $h(\cdot, \cdot)$ from $\mathcal{S}_\Delta^{(0)}(U) \times \mathcal{S}_\Delta^{(0)}(U)$ to $\Sigma_1 \times \Sigma_2$.

Lemma 3.8 *For each $u \in \mathcal{S}_\Delta^{(f)}(U)$ and $v \in \mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E)$ there exists a limit*

$$\tilde{h}(u, v) := \lim_{\varepsilon \to 0+} \int_{U_{-\varepsilon}} (u, v)_x \, dx. \qquad (3.4)$$

The corresponding pairing $\tilde{h}(\cdot, \cdot)$ is separately continuous on $\Sigma_1 \times \Sigma_2$ and $\tilde{h}(u, v) = h(u, v)$ for all $u, v \in \mathcal{S}_\Delta^{(2)}(U)$.

Proof. Using Lemma 2.25 we see that there exists a unique $w \in C^\infty(\overline{U}, E)$ satisfying

$$\begin{cases} \Delta w = v & \text{in} \quad U; \\ t(w) = 0 & \text{on} \quad \partial U. \end{cases}$$

Then

$$\begin{aligned}
\tilde{h}(u, v) &= \lim_{\varepsilon \to 0+} \int_{U_{-\varepsilon}} (u, v)_x \, dx \\
&= -\lim_{\varepsilon \to 0+} \int_{\partial U_{-\varepsilon}} G_\Delta(*_E w, u) \\
&= \lim_{\varepsilon \to 0+} \left(\int_{\partial U_{-\varepsilon}} (t(u), n(Aw))_x \, ds_{-\varepsilon} - \int_{\partial U_{-\varepsilon}} (n(Au), t(w))_x \, ds_{-\varepsilon} \right) \\
&= \int_{\partial U} (t(u), n(Aw))_x \, ds,
\end{aligned}$$

the last equality being due to the existence of weak boundary values $t(u)$ and $n(Au)$ on ∂U, cf. [ST95].

The separate continuity of the pairing $\tilde{h}(\cdot, \cdot)$ follows from (3.5) and Lemma 2.25.

Finally, the restriction of $\tilde{h}(u, v)$ to $u \in \mathcal{S}_\Delta^{(0)}(U)$ and $v \in \mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E)$ coincides with $h(u, v)$ since the Lebesgue integral is a continuous function of measurable sets.

$\square$

It is well known that $\mathcal{S}_\Delta^{(0)}(U)$ is a separable Hilbert space with a reproducing kernel (see, for instance, [Tar95a]).

As the Unique Continuation Property holds for Δ in a neighbourhood of $\overline{U}$, the Dirichlet problems for both Δ and Δ^2 in U are uniquely solvable and their Hodge parametrices G_U and $G_{\Delta^2, U}$ are, in fact, Green functions.

Lemma 3.9 *Let $K(\cdot, \cdot)$ be the reproducing kernel of $\mathcal{S}_\Delta^{(0)}(U)$. Then for all $(x, y) \in U \times U$ we have*

$$K(x, y) = -\Delta_y \int_{\partial U} \left(n(A K_{G_U}(\cdot, x)), t(\Delta K_{G_{\Delta^2, U}}(\cdot, y)) \right)_z ds. \qquad (3.5)$$

Moreover, for every fixed $x \in U$, the columns of the matrix $K(x, \cdot)$ belong to $\mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E)$.

Proof. Given any fixed $x \in U$, we solve the Dirichlet problem for Δ^2 with data

$$\begin{cases} \Delta^2 V(x, \cdot) &= \ 0 & \text{in} \ \ U, \\ t(V(x, \cdot)) &= \ 0 & \text{on} \ \ \partial U, \\ n(A V(x, \cdot)) &= \ -n(A K_{G_U}(\cdot, x)) & \text{on} \ \ \partial U. \end{cases} \qquad (3.6)$$

By the Poisson formula (2.24), for every solution $u \in \mathcal{S}_\Delta^{(0)}(U)$ and each $x \in U$ we have

$$\begin{aligned} u(x) &= \ -\int_{\partial U} (t(u), n(A K_{G_U}(\cdot, x)))_y \, ds \\ &= \ \int_{\partial U} G_\Delta(*_E V(x, \cdot), u) \\ &= \ -\int_U (u, \Delta V(x, \cdot))_y \, dy \end{aligned}$$

whence

$$0 = \int_U (u, \Delta V(x, \cdot) + K(x, \cdot))_y \, dy.$$

Since the columns of $\Delta V(x, \cdot) + K(x, \cdot)$ belong to $\mathcal{S}_\Delta^{(0)}(U)$ we deduce readily that

$$\Delta_y V(x, y) = -K(x, y)$$

for all $x, y \in U$. Representing $V(x, y)$ in $y \in U$ by the Poisson formula for Δ^2 and using (3.6), we arrive at (3.5).

For each $x, y \in U$, the integral on the right-hand side of (3.5) is well defined, and it belongs to $\mathcal{S}_\Delta(U)$ in y for every fixed $x \in U$. By the elliptic regularity, it is actually in $E_x^* \otimes \left(\mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E) \right)$, as desired.

$\square$

Theorem 3.10 *The mapping* $\mathcal{J} : \ \mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E) \to \mathcal{S}_\Delta^{(f)}(U)'$ *induced by (3.4) is a topological isomorphism of these spaces.*

Proof. Since the natural inclusion i_2 is injective the mapping $\mathcal{J}$ is injective, too (see Lemma 2.2).

By Theorem 2.3, to prove the surjectivity of the mapping $\mathcal{J}$ we need to show that the reproducing kernel $K(\cdot, \cdot)$ of the space $\mathcal{S}_\Delta^{(0)}(U)$ has the following property:

for every $\mathcal{F} \in \mathcal{S}_\Delta^{(f)}(U)'$, the section $x \mapsto *_E^{-1}\langle \mathcal{F}, K(x, \cdot)\rangle$ belongs to $\mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E)$.

Fix any $\mathcal{F} \in \mathcal{S}_\Delta^{(f)}(U)'$. Since the space $\mathcal{S}_\Delta^{(f)}(U)$ is the inductive limit of the sequence $\mathcal{S}_\Delta^{(-s)}(U)$, $s \in \mathbb{N}$, it can be specified as a subspace of $\mathcal{D}'_{\overline{U}}(X, E)$. By the Hahn-Banach Theorem, there is a section $v \in C^\infty(\overline{U}, E)$ with the property that

$$\mathcal{F}(u) = \int_U (u, v)_y \, dy$$

for all $u \in \mathcal{S}_\Delta^{(f)}(U)$. In particular, we get

$$
\begin{aligned}
*_E^{-1}\langle \mathcal{F}, K(x, \cdot)\rangle &= *_E^{-1} \int_U (K(x, \cdot), v)_y \, dy \\
&= \int_U (v, K(x, \cdot))_y \, dy \\
&= \int_U (\Delta\, G_U v, K(x, \cdot))_y \, dy \\
&= \int_{\partial U} G_\Delta \left(*_E K(x, \cdot), G_U v\right) \\
&= \int_{\partial U} (n(A\, G_U v), t(K(x, \cdot)))_y \, ds
\end{aligned}
$$

if $x \in U$.

Obviously, the integral on the right-hand side of this formula lies in $\mathcal{S}_\Delta(U)$. As $v \in C^\infty(\overline{U}, E)$ we see that $n(A\, G_U v)$ belongs to $\oplus_{j=0}^{m-1} C^\infty(\partial U, F_j)$. From Lemma 3.9 and the regularity properties of the Poisson kernels $K_{G_U}(\cdot, \cdot)$ and $K_{G_{\Delta^2, U}}(\cdot, \cdot)$ it follows that the section $*_E^{-1}\langle \mathcal{F}, K(x, \cdot)\rangle$ is C^∞ up to the boundary of U, as desired. $\qquad\square$

Since we have a commutative diagram

$$
\begin{array}{ccccc}
\mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E) & \overset{i_2}{\hookrightarrow} & \mathcal{S}_\Delta^{(0)}(U) & \overset{i_1}{\hookrightarrow} & \mathcal{S}_\Delta^{(f)}(U) \\
\cup & & \cup & & \cup \\
\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E) & \overset{i_2}{\hookrightarrow} & \mathcal{S}_A^{(0)}(U) & \overset{i_1}{\hookrightarrow} & \mathcal{S}_A^{(f)}(U)
\end{array}
$$

the pairing $\tilde{h}(\cdot,\cdot)$ induces a continuous mapping

$$\mathfrak{J} : \ \mathcal{S}_A(U) \cap C^\infty(\overline{U}, E) \to \mathcal{S}_A^{(f)}(U)'$$

which is the restriction of $\mathcal{J}$.

Write π for the Bergman projection

$$\pi : \ \mathcal{S}_\Delta^{(0)}(U) \to \mathcal{S}_A^{(0)}(U).$$

Corollary 3.11 *The mapping $\mathfrak{J}$ is a topological isomorphism of the space* $\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E)$ *onto* $\mathcal{S}_A^{(f)}(U)'$ *if and only if*

1) $\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E)$ *is dense in* $\mathcal{S}_A^{(f)}(U)$;

2) π *maps* $\mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E)$ *continuously into* $\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E)$.

Proof. According to the general scheme we put

$$\begin{aligned}
\mathfrak{S}_1 &= \mathcal{S}_A^{(f)}(U), \\
\mathfrak{S}_2 &= \mathcal{S}_A(U) \cap C^\infty(\overline{U}, E), \\
\mathfrak{V} &= \mathcal{S}_A^{(0)}(U).
\end{aligned}$$

Then the statement follows from Corollary 2.7.

$\square$

Example 3.12 Let A be a determined elliptic operator. In this case the complex $\{E^i, A^i\}$ has only one nonzero operator, which is $A^0 = A$. It is easy to check that the Neumann problem for this complex, cf. Section 2.3, is just the Dirichlet problem in $L^2(U, F)$ for the Laplacian AA^* with the Dirichlet data $n(\cdot)$ instead of $t(\cdot)$.

Suppose the Unique Continuation Property holds for the formal adjoint A^*. Then the Neumann operator N is the Green function $G_{AA^*,U}$ of the Dirichlet problem.

The elliptic regularity of the Dirichlet problem now implies that N maps $C^\infty(\overline{U}, F)$ continuously to $C^\infty(\overline{U}, F)$. Therefore, $\pi = I - A^*NA$ is a continuous mapping of $C^\infty(\overline{U}, E)$ to itself.

Further, by the Runge theorem for determined elliptic operators, cf. [Tar97, 4.1.9], $\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E)$ is dense in $\mathcal{S}_A^{(f)}(U)$.

Hence, according to Corollary 3.11, the mapping $\mathfrak{J}$ is a topological isomorphism of $\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E)$ onto $\mathcal{S}_A^{(f)}(U)'$. By reflexivity, the transposed mapping $\mathfrak{J}'$ is a topological isomorphism of the dual to $\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E)$ onto $\mathcal{S}_A^{(f)}(U)$.

Example 3.13 Let A be a column of scalar differential operators with constant coefficients in $\mathbb{R}^n$. Then A can be included into a Hilbert compatibility complex of differential operators with constant coefficients $\{E^i, A^i\}$. Under familiar hypotheses on A the compatibility complex is simply a so-called Koszul complex, cf. [Tar95a, 1.2.8].

If $U \subset\subset \mathbb{R}^n$ is a strictly convex domain with smooth boundary then the Neumann problem for $V^i = L^2(U, E^i)$ is solvable at step 1. Moreover, the Neumann operator N preserves sections from $C^\infty(U, F)$, cf. *ibid.*

For any convex domain $U \subset\subset \mathbb{R}^n$, the space $\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E)$ is known to be dense in $\mathcal{S}_A^{(f)}(U)$. By Corollary 3.11, the mapping $\mathfrak{J}$ is a topological isomorphism of $\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E)$ onto $\mathcal{S}_A^{(f)}(U)'$. The transposed mapping $\mathfrak{J}'$ then gives a topological isomorphism of the dual to $\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E)$ onto $\mathcal{S}_A^{(f)}(U)$.

3.3   Grothendieck duality

Suppose U is an open subset of $\overset{\circ}{X}$ with C^∞ boundary, such that $\partial X \cap \overline{U} = \emptyset$ or $\partial X \cap \overline{U} = \partial X$. Set

$$\Sigma_1 = \frac{\mathcal{S}_\Delta^{(f)}(U)}{\mathcal{H}(X)},$$

$$\Sigma_2 = \frac{\{u \in \mathcal{S}_\Delta(\overset{\circ}{X} \setminus \overline{U}) \cap C^\infty(X \setminus U, E) : \; t(u) = 0 \text{ on } \partial X \cap \overline{U}\}}{\mathcal{H}(X)}$$

and

$$H^m((U, \partial X), E) = \{u \in H^m(U, E) : \; t(u) = 0 \text{ on } \partial X \cap \overline{U}\}.$$

For $u \in H^m(U, E)$, we define

$$\begin{aligned}
\mathcal{E}_U(u) &\in \overset{\circ}{H}{}^m(X, E) \\
&= H^m((X, \partial X), E)
\end{aligned}$$

by

$$\left\{\begin{aligned}
\mathcal{E}_U(u) &= u & \text{in} \quad & U; \\
\Delta \mathcal{E}_U(u) &= 0 & \text{in} \quad & X \setminus \overline{U}; \\
t(\mathcal{E}_U(u)) &= t(u) & \text{on} \quad & \partial U.
\end{aligned}\right.$$

Since A possesses the Unique Continuation Property in $\overset{\circ}{X}$ the section $\mathcal{E}_U(u)$ is uniquely determined for every $u \in H^m((U, \partial X), E)$.

Write H for the Hodge projection onto $\mathcal{H}(X)$, cf. Theorem 2.26. It was proved in [SST00] that the Hermitean form

$$h_U(u, v) = \int_X (A\mathcal{E}_U(u), A\mathcal{E}_U(v))_x \, dx + \int_X (H\mathcal{E}_U(u), H\mathcal{E}_U(v))_x \, dx \qquad (3.7)$$

is a scalar product on $H^m((U, \partial X), E)$ which induces a topology equivalent to the original one.

Obviously, $\mathcal{H}(X)$ is a closed subspace of

$$\mathcal{S}_\Delta^{(m)}(X \setminus \overline{U}, \partial X) := \mathcal{S}_\Delta(\overset{\circ}{X} \setminus \overline{U}) \cap H^m((X \setminus \overline{U}, \partial X), E),$$

and we put

$$V = \mathcal{S}_\Delta^{(m)}(X \setminus \overline{U}, \partial X) \ominus \mathcal{H}(X),$$

"$\ominus$" meaning the orthogonal complement with respect to the scalar product $h_{X \setminus \overline{U}}(\cdot, \cdot)$.

Lemma 3.14 *As defined above, V is a Hilbert space with reproducing kernel. If $\mathcal{H}(X) = 0$ then this kernel is given by*

$$K(x, \cdot) = \mathcal{E}_U *_E^{-1} K_G(x, \cdot)$$

for every $x \in X \setminus \overline{U}$, where G is the Hodge parametrix for the Dirichlet problem in X.

Proof. The space under study is a closed subspace of $\mathcal{S}_\Delta^{(m)}(\overset{\circ}{X} \setminus \overline{U})$, hence it is a Hilbert space with reproducing kernel.

If, in addition, $\mathcal{H}(X) = 0$ then G is a two-sided fundamental solution of Δ in the interior of X. It follows that $\Delta \mathcal{E}_U K_G(\cdot, x) = 0$ away from ∂U, for every fixed $x \in X \setminus \overline{U}$.

By the definition of a reproducing kernel, we get

$$u(x) = h_{X \setminus \overline{U}}(u, K(x, \cdot)) \tag{3.8}$$

in $\overset{\circ}{X} \setminus \overline{U}$ for all $u \in V$. On the other hand, if $x \in \overset{\circ}{X} \setminus \overline{U}$ then by Stokes' formula we have

$$\begin{aligned}
u(x) &= \int_{\partial U} G_\Delta(K_G(x, \cdot), u) \\
&= \int_{\partial U} \left(t(u), n(A *_E^{-1} K_G(x, \cdot)) \right)_y ds - \int_{\partial U} \left(n(Au), t(*_E^{-1} K_G(x, \cdot)) \right)_y ds \\
&= h_{X \setminus \overline{U}}(u, \mathcal{E}_U *_E^{-1} K_G(x, \cdot)),
\end{aligned}$$

which gives the desired conclusion when combined with (3.8).

$\square$

Define the mapping $i_2 : \Sigma_2 \to V$ in the following way. Pick an element $[u] \in \Sigma_2$, $[u]$ being the equivalence class of any $u \in \mathcal{S}_\Delta^{(\infty)}(X \setminus \overline{U})$ satisfying $t(u) = 0$ on $\partial X \cap \overline{U}$. Obviously, $u \in \mathcal{S}_\Delta^{(m)}(X \setminus \overline{U}, \partial X)$. We set $i_2[u]$ to be the

orthogonal projection of u to V. It is easy to check that $i_2 : \Sigma_2 \to V$ is well defined and continuous.

Further, for $u \in V$ we set $i_1 u$ to be the equivalence class in Σ_1 of the restriction of $\mathcal{E}_{X \setminus \overline{U}}(u)$ to U.

Our next task is to extend the scalar product $h_{X \setminus \overline{U}}(\cdot, \cdot)$ from $V \times V$ to $\Sigma_1 \times \Sigma_2$. Let $[u] \in \Sigma_1$ and $[v] \in \Sigma_2$. For any representative $u \in [u]$, the Cauchy data $t(u)$ and $n(Au)$ have weak boundary values on the boundary of U, cf. [ST95]. We set

$$\tilde{h}_{X \setminus \overline{U}}\left([u], [v]\right) = - \int_{\partial U} G_\Delta(*_E v, u) \tag{3.9}$$

where $v \in [v]$.

Lemma 3.15 *The pairing (3.9) does not depend on the particular choice of the representatives $u \in [u]$ and $v \in [v]$. Moreover, it is separately continuous and*

$$\tilde{h}_{X \setminus \overline{U}}\left(i_1 u, [v]\right) = h_{X \setminus \overline{U}}\left(u, i_2[v]\right)$$

for all $u \in V$ and $[v] \in \Sigma_2$.

Proof. By Stokes' formula, we get

$$\begin{aligned}
\int_{\partial U} G_\Delta(*_E v, u) &= \int_U (\Delta u, v)_x dx - \int_U (u, \Delta v)_x dx \\
&= 0
\end{aligned}$$

for $v \in \mathcal{H}(X)$. Similarly

$$\begin{aligned}
- \int_{\partial U} G_\Delta(*_E v, u) &= \int_{X \setminus \overline{U}} (\Delta u, v)_x dx - \int_{X \setminus \overline{U}} (u, \Delta v)_x dx \\
&= 0
\end{aligned}$$

for any $u \in \mathcal{H}(X)$. This shows that $\tilde{h}_{X \setminus \overline{U}}(\cdot, \cdot)$ is independent of the choice of $u \in [u]$ and $v \in [v]$.

By the above, the convergence of a sequence $\{u_\nu\}$ in $\mathcal{S}_\Delta^{(f)}(U)$ implies the convergence of both $\{t(u_\nu)\}$ and $\{n(Au_\nu)\}$ in $\oplus_{j=0}^{m-1} \mathcal{D}'(\partial U, F_j)$. Therefore, $\tilde{h}_{X \setminus \overline{U}}(\cdot, \cdot)$ is separately continuous provided the spaces Σ_1 and Σ_2 are endowed with the quotient topology.

Since the solutions of V have finite order of growth close to ∂U it follows that

$$\begin{aligned}
\tilde{h}_{X \setminus \overline{U}}\left(i_1 u, [v]\right) &= - \int_{\partial U} G_\Delta\left(*_E v, \mathcal{E}_{X \setminus \overline{U}} u\,|_U\right) \\
&= \int_{\partial U} \left((n(A \mathcal{E}_{X \setminus \overline{U}} u\,|_U), t(v))_y - (t(\mathcal{E}_{X \setminus \overline{U}} u\,|_U), n(Av))_y \right) ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\partial U} \left(n(A\mathcal{E}_{X\setminus\overline{U}} u\,|_U), t(\mathcal{E}_{X\setminus\overline{U}} v) \right)_y ds - \int_{\partial U} \left(t(u), \nu(Av) \right)_y ds \\
&= h_{X\setminus\overline{U}}(u, v) - \int_X \left(H\mathcal{E}_{X\setminus\overline{U}} u, H\mathcal{E}_{X\setminus\overline{U}} v \right)_x dx \\
&= h_{X\setminus\overline{U}}(u, v),
\end{aligned}
$$

the last equality being a consequence of the fact that w is orthogonal to $\mathcal{H}(X)$ with respect to $h_{X\setminus\overline{U}}(\cdot,\cdot)$. The right-hand side here is obviously equal to $h_{X\setminus\overline{U}}(u, i_2[v])$, which completes the proof. $\square$

This lemma gives rise to a mapping $\mathcal{J} : \Sigma_2 \to \Sigma_1'$ induced by the pairing $h_{X\setminus\overline{U}}(\cdot,\cdot)$.

Theorem 3.16 *The mapping*

$$
\mathcal{J} : \ \frac{\mathcal{S}_\Delta^{(\infty)}(\overset{\circ}{X} \setminus \overline{U}, \partial X)}{\mathcal{H}(X)} \to \left(\frac{\mathcal{S}_\Delta^{(f)}(U)}{\mathcal{H}(X)} \right)'
$$

induced by (3.9) is a topological isomorphism of these spaces.

Proof. By the very construction, the mapping i_2 is one-to-one. Lemma 2.2 shows that $\mathcal{J}$ is one-to-one, too.

Let us prove the surjectivity of $\mathcal{J}$. To this end, pick a continuous linear functional $\mathcal{F}$ on Σ_1. Since $\mathcal{H}(X)$ is finite dimensional, $\mathcal{F}$ can be specified to a continuous linear functional on $\mathcal{S}_\Delta^{(f)}(U)$ vanishing on $\mathcal{H}(X)$. By Theorem 3.10, there is a section $w \in \mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E)$ orthogonal to $\mathcal{H}(X)$, with the property that

$$
\langle \mathcal{F}, [u] \rangle = \int_U (u, w)_x \, dx \tag{3.10}
$$

for all $[u] \in \Sigma_2$, u being a representative of $[u]$.

Set

$$
\begin{aligned}
v(y) &= *_E^{-1} \int_U (K_G(\cdot, y), w)_x \, dx \\
&= G\left(\chi_U w\right)(y)
\end{aligned}
$$

for $y \in \overset{\circ}{X}$, where χ_U is the characteristic function of U. By Theorem 2.26 we deduce that $v \in H^{2m}(X, E)$.

Further, G is a Hodge parametrix whence $t(v) = 0$ on ∂X. We assert that $\Delta v = 0$ away from $\overline{U}$ in the interior of X.

Indeed, let $u \in C^\infty(X, E)$ have compact support in $\mathring{X} \setminus \overline{U}$. Then

$$
\begin{aligned}
\int_{X \setminus \overline{U}} (\Delta u, v)_y \, dy &= \int_X (\Delta u, v)_y \, dy \\
&= \int_U (G \Delta u, w)_x \, dx \\
&= \int_U (u, w)_x \, dx - \int_U (H u, w)_x \, dx \\
&= 0,
\end{aligned}
$$

for $u \equiv 0$ in U and w is orthogonal to $\mathcal{H}(X)$.

Since G has the transmission property with respect to ∂U, we see that $v \in C^\infty(\overline{U}, E)$ whence $t(v) \in \oplus_{j=0}^{m-1} C^\infty(\partial U, F_j)$. The elliptic regularity of the Dirichlet problem now yields $v \in C^\infty(X \setminus U, E)$, and so u determines an equivalence class $[v] \in \Sigma_2$.

If $u \in \mathcal{S}_\Delta^{(f)}(U)$ then

$$
u(x) = - \lim_{\varepsilon \to 0+} \int_{\partial U_{-\varepsilon}} G_\Delta(K_G(x, \cdot), u) + \lim_{\varepsilon \to 0+} H\left(\chi_{U_{-\varepsilon}} u\right)(x)
$$

for all $x \in U$. Hence it follows that

$$
\begin{aligned}
\langle \mathcal{F}, [u] \rangle &= \int_U (u, w)_x \, dx \\
&= - \lim_{\varepsilon \to 0+} \int_{\partial U_{-\varepsilon}} G_\Delta(*_E v, u) + \lim_{\varepsilon \to 0+} \int_U \left(H\left(\chi_{U_{-\varepsilon}} u\right), w\right)_x \, dx \\
&= - \lim_{\varepsilon \to 0+} \int_{\partial U_{-\varepsilon}} G_\Delta(*_E v, u) \\
&= \tilde{h}_{X \setminus \overline{U}}(u, v)
\end{aligned}
$$

because w is orthogonal to $\mathcal{H}(X)$.

Finally, the topological arguments are the same as in Theorem 3.2, which completes the proof.

$\square$

3.4   Pairing in Sobolev spaces

Set

$$
\begin{aligned}
\Sigma_1 &= \frac{\mathcal{S}_\Delta^{(f)}(U)}{\mathcal{H}(X)}, \\
\Sigma_2 &= \frac{\mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E)}{\mathcal{H}(X)}
\end{aligned}
$$

and

$$V = \mathcal{S}_\Delta^{(m)}(U) \ominus \mathcal{H}(X),$$

"$\ominus$" meaning the orthogonal complement with respect to the scalar product $h_U(\cdot, \cdot)$. When equipped with the scalar product $h_U(\cdot, \cdot)$, the space V is obviously Hilbert.

For $[u] \in \Sigma_2$ we set

$$E_U([u]) = [\mathcal{E}_U(u)\,|_{X \setminus \overline{U}}],$$

the right-hand side being an equivalence class in $\mathcal{S}_\Delta^{(\infty)}(X \setminus \overline{U}, \partial X) / \mathcal{H}(X)$. This class is independent of which representative u of $[u]$ we choose to define it. Indeed, from the Unique Continuation Property for Δ and the elliptic regularity of the Dirichlet Problem it follows that $\mathcal{E}_U$ induces a topological isomorphism of $\mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E)$ onto $\mathcal{S}_\Delta^{(\infty)}(X \setminus \overline{U}, \partial X)$. In particular, if $u \in \mathcal{H}(X)$ then $\mathcal{E}_U(u) = u$ in the complement of $\overline{U}$. This gives us the desired independence, hence E_U is well defined.

Since the space $\mathcal{H}(X)$ is finite dimensional we immediately obtain the following lemma.

Lemma 3.17 *The mapping E_U is a topological isomorphism*

$$\frac{\mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E)}{\mathcal{H}(X)} \;\overset{\cong}{\to}\; \frac{\mathcal{S}_\Delta^{(\infty)}(X \setminus \overline{U}, \partial X)}{\mathcal{H}(X)}.$$

Combining Lemma 3.17 and Theorem 3.16 we see that Σ_2 and Σ_1' are topologically isomorphic. However, we want to recover this result within the general scheme of Section 2.

To this end, we write $i_1 : V \to \Sigma_1$ and $i_2 : \Sigma_2 \to V$ for the natural embeddings. They are obviously injective and continuous. We define an extension $\tilde{h}_U(\cdot, \cdot)$ of $h_U(\cdot, \cdot)$ by

$$\tilde{h}_U([u], [v]) = \tilde{h}_{X \setminus \overline{U}}([u], E_U([v])). \tag{3.11}$$

Lemma 3.18 *As defined by (3.11), the pairing $\tilde{h}_U(\cdot, \cdot)$ does not depend on the choice of $u \in [u]$ and $v \in [v]$. Moreover, it is separately continuous and satisfies*

$$\tilde{h}_U(i_1 u, [v]) = h_U(u, i_2[v])$$

for all $u \in V$ and $[v] \in \Sigma_2$.

Proof. The pairing is independent of the choice of $u \in [u]$ and $v \in [v]$ because so are the pairing $\tilde{h}_{X \setminus \overline{U}}$ and the mapping E_U.

Furthermore, from the definition of $\mathcal{E}_U$ it follows that $\mathcal{E}_{X\setminus\overline{U}}\mathcal{E}_U = \mathcal{E}_U$ on $\mathcal{S}_\Delta^{(m)}(U)$. Hence, by Lemma 3.15 we get

$$
\begin{aligned}
\tilde{h}_U\left(i_1 u, [v]\right) &= \tilde{h}_{X\setminus\overline{U}}\left(i_1\mathcal{E}_{X\setminus\overline{U}}\mathcal{E}_U(u), E_U([v])\right) \\
&= h_{X\setminus\overline{U}}\left(\mathcal{E}_{X\setminus\overline{U}}\mathcal{E}_U(u), i_2 E_U([v])\right) \\
&= h_U\left(u, i_2[v]\right),
\end{aligned}
$$

as desired. $\square$

Theorem 3.19 *The mapping*

$$
\mathcal{J}:\ \frac{\mathcal{S}_\Delta(U)\cap C^\infty(\overline{U}, E)}{\mathcal{H}(X)} \to \left(\frac{\mathcal{S}_\Delta^{(f)}(U)}{\mathcal{H}(X)}\right)'
$$

induced by (3.11) is a topological isomorphism of these spaces.

Proof. As mentioned, this assertion follows from Lemma 3.17 and Theorem 3.16.

$\square$

As we have a commutative diagram

$$
\begin{array}{ccccc}
\dfrac{\mathcal{S}_\Delta(U)\cap C^\infty(\overline{U}, E)}{\mathcal{H}(X)} & \xhookrightarrow{i_2} & \mathcal{S}_\Delta^{(m)}(U)\ominus\mathcal{H}(X) & \xhookrightarrow{i_1} & \dfrac{\mathcal{S}_\Delta^{(f)}(U)}{\mathcal{H}(X)} \\
\cup & & \cup & & \cup \\
\dfrac{\mathcal{S}_A(U)\cap C^\infty(\overline{U}, E)}{\mathcal{H}(X)} & \xhookrightarrow{i_2} & \mathcal{S}_A^{(m)}(U)\ominus\mathcal{H}(X) & \xhookrightarrow{i_1} & \dfrac{\mathcal{S}_A^{(f)}(U)}{\mathcal{H}(X)}
\end{array}
$$

the pairing $\tilde{h}_U(\cdot,\cdot)$ induces a continuous mapping

$$
\mathfrak{J}:\ \frac{\mathcal{S}_A(U)\cap C^\infty(\overline{U}, E)}{\mathcal{H}(X)} \to \left(\frac{\mathcal{S}_A^{(f)}(U)}{\mathcal{H}(X)}\right)' \tag{3.12}
$$

which is the restriction of $\mathcal{J}$.

Write π for the h_U-orthogonal projection

$$
\pi:\ \mathcal{S}_\Delta^{(m)}(U)\ominus\mathcal{H}(X) \to \mathcal{S}_A^{(m)}(U)\ominus\mathcal{H}(X).
$$

Corollary 3.20 *The mapping $\mathfrak{J}$ is a topological isomorphism of the spaces (3.12) if and only if*

1) $\mathcal{S}_A(U)\cap C^\infty(\overline{U}, E)$ *is dense in* $\mathcal{S}_A^{(f)}(U)$;
2) π *maps* $\mathcal{S}_\Delta(U)\cap C^\infty(\overline{U}, E)$ *continuously into* $\mathcal{S}_A(U)\cap C^\infty(\overline{U}, E)$.

Proof. According to the general scheme we put

$$
\mathfrak{S}_1 = \frac{\mathcal{S}_A^{(f)}(U)}{\mathcal{H}(X)},
$$
$$
\mathfrak{S}_2 = \frac{\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E)}{\mathcal{H}(X)},
$$
$$
\mathfrak{V} = \mathcal{S}_A^{(m)}(U) \ominus \mathcal{H}(X).
$$

Then the statement follows from Corollary 2.7.

$\square$

We thus see that Corollary 3.20 is formulated in just the same way as Corollary 3.11, the only difference being in the pairing $\tilde{h}$ and corresponding orthogonal projection π.

Note that the projection π can be described as the limit of iterations of double layer potentials, cf. [NS96, SST00]. We next make use of this description in order to find several cases where the conditions 1) and 2) of Corollary 3.20 are fulfilled.

Example 3.21 Let A be a determined elliptic homogeneous operator of order $m < n/2$ with constant coefficients in $X = \mathbb{R}^n$. Then the Laplacian $\Delta = A^*A$ has a two-sided fundamental solution of convolution type vanishing at infinity (see for instance [Tar95b]). By the Liouville Theorem we deduce that $\mathcal{H}(X) = \{0\}$.

Since Δ is formally selfadjoint, there is a formally selfadjoint fundamental solution of convolution type, say G. It was proved in [SST00] that

$$
\pi = \lim_{N \to \infty} M^N
$$

where

$$
(Mu)(x) = -\int_{\partial U} G_A(K_{GA^*}(x - \cdot), u)
$$

for all $u \in H^m(U, E)$, the limit being in the strong operator topology of $H^m(U, E)$.

We can certainly assume that G is of the form $G = \operatorname{ad} A(D) \operatorname{ad} A^*(D) g$ where $\operatorname{ad} A(D)$ is the inverse array of $A(D)$, and g a fundamental solution of the (scalar) operator $\det \Delta(D) = |\det A(D)|^2$. It is easy to verify that in this particular case the kernel $K_{GA^*}(x - y)$ gives a two-sided fundamental solution for A. Hence $M^2 = M$ and $\pi = M$.

Finally, as GA^* has the transmission property with respect to any smooth hypersurface in $\mathbb{R}^n$ we conclude that M maps $C^\infty(\overline{U}, E)$ continuously into $\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E)$.

Example 3.22 Let A be a determined elliptic operator as in Example 3.12, and let A^* possess the Unique Continuation Property in the interior of X. Then the condition 1) of Corollary 3.20 is fulfilled.

We are going to invoke the Neumann operator for the short complex consisting of $A_0 = A$ only, cf. Section 2.3, in order to prove that the condition 2) of Corollary 3.20 also holds.

Consider the continuous operator $T \colon H^m(U, E) \to L^2(U, F)$ induced by A, i.e., $Tu = Au$. According to Theorem 6.2 from [SST00] the adjoint T^* of T with respect to $h_U(\cdot, \cdot)$ is given by $T_U f = GA^*(\chi_U f)$ where A^* is the formal adjoint of A.

As we are interested in the projection to $\mathcal{S}_A^{(m)}(U) \ominus \mathcal{H}(X)$ we need to identify the adjoint of the restriction of T to $H^m(U, E) \ominus \mathcal{H}(X)$. Obviously, this adjoint is equal to $(1 - \pi_{\mathcal{H}(X)})T^*$ where $\pi_{\mathcal{H}(X)}$ is the orthogonal projection onto $\mathcal{H}(X)$ with respect to $h_U(\cdot, \cdot)$.

We have $\pi_{\mathcal{H}(X)} = H\mathcal{E}_U$ where H is the $L^2(X, E)$-orthogonal projection to $\mathcal{H}(X)$. Since $\Delta(T_U f) = A^*(\chi_U f)$ in the sense of distributions in the interior of X, we get $\mathcal{E}_U T_U = T_U$ on all of X. It follows that

$$
\begin{aligned}
\left(1 - \pi_{\mathcal{H}(X)}\right) T^* &= \left(1 - H\mathcal{E}_U\right) T_U \\
&= \left(1 - H\right) T_U .
\end{aligned}
$$

The Laplacian $L^1 = TT^*$ is a bounded selfadjoint operator in $L^2(U, F)$. Let us show that it is injective.

Indeed, $L^1 f = 0$ if and only if $T_U f = 0$, for $T^* = T_U$. Moreover, if $T_U f = 0$ in U then $T_U f = E_U T_U f = 0$ in X, and so $\Delta(T_U f) = A^*(\chi_U f) = 0$ in X. As A^* possesses the Unique Continuation Property in a neighbourhood of U we see that $\chi_U f \equiv 0$ in U.

Since L^1 is selfadjoint we conclude that the range of TT_U is dense in $L^2(U, F)$.

Moreover, the range of T is equal to $L^2(U, F)$, and so the range of $T_U = T^*$ is closed in $H^m(U, E)$, too. It follows that

$$
H^m(U, E) = \ker T \oplus T_U L^2(U, F),
$$

hence the range of L^1 coincides with that of T. We have thus proved that L^1 is an isomorphism of $L^2(U, F)$. In fact, $L^1 = 1$ in the case considered in Example 3.21.

We next show that $T_U (L^1)^{-1} T$ maps $C^\infty(\overline{U}, E)$ continuously to $C^\infty(\overline{U}, E)$, and hence π does so. For this purpose, pick $g \in L^2(U, F)$. Then there exists $f \in L^2(U, F)$ satisfying $TT_U f = g$ in U.

Note that $AT_U f$ is defined on all of X and belongs to $L^2(X, F)$. Therefore, g can be extended to a section $\tilde{g} \in L^2(X, F)$ in such a way that $AT_U f = \tilde{g}$ in

X. It follows from Theorem 2.26 that

$$
\begin{aligned}
A^* \tilde{g} &= A^* A \, T_U f \\
&= (1 - H) \, A^* \left(\chi_U f \right) \\
&= A^* \left(\chi_U f \right),
\end{aligned}
\tag{3.13}
$$

in $H^{-m}(X, E)$, for $H A^* = 0$.

This formula implies, in particular, that $A A^* \tilde{g} = 0$ away from $\overline{U}$ in the interior of X, and thus $\tilde{g}$ is C^∞ on this set.

Since A^* (as well as A) possesses the Unique Continuation Property in $\overset{\circ}{X}$, there exists a two-sided fundamental solution to this operator, say Φ_{A^*} in a neighbourhood O of $\overline{U}$. Applying it to both sides of formula (3.13) yields for each $x \in U$

$$
\left(L^1 \right)^{-1} g\left(x \right) = \tilde{g}(x) + \int_{\partial O} G_{A^*} \left(K_{\Phi_{A^*}} (x, \cdot), \tilde{g} \right).
\tag{3.14}
$$

It is well known that the kernel of Φ_{A^*} is smooth outside of the diagonal $\{ x = y \}$. Therefore, the boundary integral in (3.14) is a C^∞ section of F near $\overline{U}$.

We thus deduce that the inverse $\left(L^1 \right)^{-1}$ preserves the C^∞ sections of F over $\overline{U}$.

Finally, if $u \in C^\infty(\overline{U}, E)$ then $(L^1)^{-1} A u \in C^\infty(\overline{U}, F)$, and so $T_U \left(L^1 \right)^{-1} A u$ belongs to $C^\infty(\overline{U}, E)$ because the Green operator G has the transmission property with respect to ∂U. The continuity of π now follows from the equality $\pi = I - T_U \left(L^1 \right)^{-1} T$ modulo the smoothing operator $H \, T_U \left(L^1 \right)^{-1} T$. Thus, conditions 1) and 2) of Corollary 3.20 hold for A.

Example 3.23 Let A be included to an elliptic complex of differential operators $A^i \in \mathrm{Diff}^m(X; E^i, E^{i+1})$ of the same order on X, so that $A^0 = A$. We formulate a particular Neumann problem, cf. Section 2.3, corresponding to our situation.

We have a continuous operator $A \colon H^m(U, E) \ominus \mathcal{H}(X) \to L^2(U, F)$. Arguing as in Example 3.22 we see that the adjoint A^* of A with respect to $h_U(\cdot, \cdot)$ is given by $T_U f = G A^* \left(\chi_U f \right)$.

Let n_{A^2} represent the Cauchy data with respect the formal adjoint of A^2 and

$$
H^m_{n_A}(U, E^2) = \{ u \in H^m(U, E^2) : \ n_A(u) = 0 \}
$$

on ∂U. Obviously, it is a closed subspace of $H^m(U, E^2)$.

For $x \in U$, pick a cut-off function ω_x at x, i.e., any C^∞ function with a compact support in U, equal to 1 near x and vanishing outside of a larger neighbourhood of x. The difference $1 - \omega_x$ is equal to 1 close to ∂U, hence for

every $w \in H^m_{\nu_A}(U, E^2)$ we get

$$
\begin{aligned}
T_U \left(A^1 \right)^* w &= G \left(\omega_x + (1 - \omega_x) \right) A^* (\chi_U A^{1*} w) \\
&= (G \omega_x) A^* A^{1*} w + (G(1 - \omega_x)) A^* A^{1*} (\chi_U w) \\
&= 0
\end{aligned}
$$

at x.

Consider the complex

$$
0 \longleftarrow H^m(U, E) \xleftarrow{T_U} L^2(U, E^1) \xleftarrow{A^{1*}} H^m_{\nu_A}(U, E^2) \longleftarrow \ldots, \qquad (3.15)
$$

the fragments indicated by dots being unimportant in the sequel.

Under our assumptions on $\{E^i, A^i\}$, the Laplacians $\Delta^i = A^{i*} A^i + A^{i-1} A^{i-1*}$ are elliptic operators of order $2m$.

Let B^i be the block operator

$$
B^i = \left(\begin{array}{c} A^i \\ A^{i-1*} \end{array} \right) : \; H^m(U, E^i) \rightarrow \begin{array}{c} L^2(U, E^{i+1}) \\ \oplus \\ L^2(U, E^{i-1}) \end{array}
$$

whose Laplacian is $B^{i*} B^i = \Delta^i$. As Δ^i are elliptic differential operators, the same is true for B^i.

We endow the space $H^m(U, E^2)$ with the scalar product

$$
h^2_U(f, g) = \int_X \left(B \mathcal{E}^2_U(f), B \mathcal{E}^2_U(g) \right)_x dx + \int_X \left(H \mathcal{E}^2_U(f), H \mathcal{E}^2_U(g) \right)_x dx
$$

constructed for B^2 in the same way as the scalar product $h_U = h^0_U$ on $H^m(U, E)$ was constructed for A.

Write G^2 for the Hodge parametrix of the Dirichlet problem corresponding to Δ^2 in X, and $T^{(2,2)}_U$ for the composition $G^2 A^1 \chi_U$. Then the adjoint complex to (3.15) is given by:

$$
0 \longrightarrow H^m(U, E) \xrightarrow{A} L^2(U, E^1) \xrightarrow{p \, T^{(2,2)}_U} H^m_{n_A}(U, E^2) \longrightarrow \ldots \qquad (3.16)
$$

where $p \colon H^m(U, E^2) \to H^m_{n_A}(U, E^2)$ is the orthogonal projection with respect to $h^2_U(\cdot, \cdot)$. Indeed, by Theorem 6.2 of [SST00] the adjoint B^{2*} to B^2 with respect to $h^2_U(\cdot, \cdot)$ is given by

$$
\begin{aligned}
T^2_U f &= G^2 B^{2*} (\chi_U f) \\
&= \left(G^2 A^{2*} (\chi_U f), G^2 A^1 (\chi_U f) \right) \\
&= (T^{(2,1)}_U, T^{(2,2)}_U) f
\end{aligned}
$$

where B^{2*} is the formal adjoint of B^2.

It follows that $(A^{1*})^* = T_U^{(2,2)}$ is the adjoint to A^{1*} in the sense of Hilbert spaces if A^{1*} is considered as a bounded operator from $H^m(U, E^2)$ to $L^2(U, E^1)$. Obviously, $p\, T_U^{(2,2)}$ is the adjoint in the sense of Hilbert spaces for the restriction of A^{1*} to $H_{n_A}^m(U, E^2)$. Hence the Laplacian of (3.16) at step 1 is the bounded selfadjoint operator

$$L^1 = A^{1*} p\, T_U^{(2,2)} + A T_U$$

on $L^2(U, F)$.

From now on we assume that $\mathcal{H}^2(X) = 0$. Let us show that under this assumption the null-space of $p\, T_U^{(2,2)}$ just amounts to $\mathcal{S}_{A^1}^{(m)}(U)$. Denote by G_U^2 the Green function of the Dirichlet problem for Δ^2 in U. Then for every $f \in H^m(U, E^2)$ we have

$$f = M_U^2 f + G_U^2 \Delta^2 f$$

by the Poisson formula (2.24). As $\mathcal{E}_U^2(g) = 0$ in $X \setminus \overline{U}$ for any $g \in \overset{\circ}{H}{}^m(U, E^2)$ we conclude that

$$
\begin{aligned}
h_U^2(M_U^2 f, G_U^2 \Delta^2 f) &= \int_U (B^2 M_U^2 f, B^2 G_U^2 \Delta^2 f)_x dx \\
&= \int_U (\Delta^2 M_U^2 f, G_U^2 \Delta^2 f)_x dx - \int_{\partial U} G_{B^{2*}}(*_{E^2} G_U^2 \Delta^2 f, B^2 M_U^2 f) \\
&= 0
\end{aligned}
$$

because $\Delta^2 M_U^2 f = 0$ in U and $G_U^2 \Delta^2 f \in \overset{\circ}{H}{}^m(U, E^2)$. We thus deduce that the Poisson formula gives an orthogonal decomposition with respect to $h_U^2(\cdot, \cdot)$ if $\mathcal{H}^2(X) = 0$. Since

$$\overset{\circ}{H}{}^m(U, E^2) \hookrightarrow H_{n_A}^m(U, E^2)$$

we actually see that

$$
\begin{aligned}
\left(H_{n_A}^m(U, E^2)\right)^\perp &\hookrightarrow \left(\overset{\circ}{H}{}^m(U, E^2)\right)^\perp \\
&= \mathcal{S}_{\Delta^2}^{(m)}(U).
\end{aligned}
$$

Thus, $p\, T_U^{(2,2)} f = 0$ implies $\Delta^2 T_U^{2,2} f = 0$ in U. On the other hand, in X we have

$$
\begin{aligned}
\Delta^2 T_U^{(2,2)} f &= A^1(\chi_U f) - H^2 A^1(\chi_U f) \\
&= A^1(\chi_U f)
\end{aligned}
$$

because $H^2 A^1 = 0$. In particular this means that $A^1 f = 0$ in U if and only if $\Delta^2 T_U^{(2,2)} f = 0$ in U.

Conversely, if $A^1 f = 0$ in U then $\Delta^2 T_U^{(2,2)} f = 0$ in U, and

$$T_U^{(2,2)} f(x) = \int_{\partial U} G_{B^{2*}}(K_{G^2}(x, \cdot), 0 \oplus f)$$

for all $x \in U$, the last equality being a consequence of Stokes' formula. Moreover, for all $g \in H^m_{n_A}(U, E^2)$ we have

$$
\begin{aligned}
h^2_U \left(T^{(2,2)}_U f, g \right) &= \int_X \left(B^2 T^{(2,2)}_U f, B^2 \mathcal{E}^2_U g \right)_x dx \\
&= \int_{\partial U} G_{B^2} \left(0 \oplus *_{E^1} f, g \right) \\
&= \int_{\partial U} G_{A^{1*}} \left(*_{E^1} f, g \right) \\
&= 0,
\end{aligned}
$$

the last equality being due to jump theorems for a single layer potential and the fact that $n_A(\mathcal{E}^2_U(g)) = n_A(g) = 0$ on ∂U. Hence $A^1 f = 0$ in U implies $p\, T^{(2,2)}_U f = 0$ in U, as desired.

Let us describe the null-space of L^1. Note that $L^1 f = 0$ if and only if both $T_U f$ and $p\, T^{(2,2)}_U f$ vanish, for the operator

$$
\begin{pmatrix} p\, T^{(2,2)}_U \\ T_U \end{pmatrix} : \; L^2(U, F) \to \begin{array}{c} H^m_{n_A}(U, E^2) \\ \oplus \\ H^m(U, E) \end{array}
$$

is adjoint to the differential operator B^{1*}.

Hence $\ker L^1 = \ker T_U \cap \mathcal{S}^{(0)}_{A^1}(U)$. It was proved in [SST00] (see Lemma 8.4) that

$$
\ker T_U \cap \mathcal{S}_{A^1}(U) = \left\{ f \in L^2(U, F) : \; A^* f = 0, \; A^1 f = 0, \; n(f) = 0 \text{ on } \partial U \right\}.
$$

This is the so-called "harmonic space" at step 1 for the differential complex $\{E^i, A^i\}$ in U. One usually realises it as the null-space for the Neumann problem in the L^2-setting for $\{E^i, A^i\}$.

As the Laplacian L^1 is defined everywhere on $L^2(U, F)$ the Neumann problem for the complex (3.16) at step 1 reads as follows: given any $f \in L^2(U, F)$, find $g \in L^2(U, F)$ satisfying $L^1 g = f$. If the range of L^1 is closed, then so is the range of $A \colon H^m(U, E) \to L^2(U, F)$, too. For the Dolbeault complex this latter fails to be the case even for the small balls U of $\mathbb{C}^n$. Hence for the Dolbeault complex the Neumann problem in the present setting can not highlight any properties of the projection π.

The last observation in Example 3.23 leads us to another choice of function spaces, which could be $H^m(U, E^i)$.

Example 3.24 Let $X = \mathbb{R}^n$, $n \geq 3$, $U \subset \mathbb{R}^n$ a ball, and A a Dirac operator in $\mathbb{R}^n$, cf. Example 3.5.

It is proved in [Shl96] that there are systems $\{b_\nu\}$ and $\{c_\mu\}$ of ($\mathbb{C}^k$-valued) homogeneous harmonic polynomials, such that

a) $\{b_\nu\}$ is an orthogonal basis in all the spaces $\mathcal{S}_A^{(s)}(U)$, $s \in \mathbb{Z}_+$, simultaneously, and an orthonormal basis in $\mathcal{S}_A^{(m)}(U)$ with respect to $h_U(\cdot, \cdot)$;

b) $\{b_\nu\} \cup \{c_\mu\}$ is an orthogonal basis in all the spaces $\mathcal{S}_\Delta^{(s)}(U)$, $s \in \mathbb{Z}_+$, simultaneously, and an orthonormal basis in $\mathcal{S}_\Delta^{(m)}(U)$ with respect to $h_U(\cdot, \cdot)$.

Property $a)$ implies, in particular, that condition 1) of Corollary 3.20 is fulfilled. Moreover, the projection π, if restricted to $\mathcal{S}_\Delta^{(s)}(U)$, coincides with the $h_U(\cdot, \cdot)$-orthogonal projection $\mathcal{S}_\Delta^{(s)}(U) \to \mathcal{S}_A^{(s)}(U)$. It follows that π maps $\mathcal{S}_\Delta(U) \cap C^\infty(\overline{U}, E)$ continuously into $\mathcal{S}_A(U) \cap C^\infty(\overline{U}, E)$. Hence Corollary 3.20 holds for A.

4 Duality for solutions of arbitrary order of growth

In this section we will describe the dual space of $\mathcal{S}_A(U)$ by using various pairings in Hilbert spaces of solutions to the system $Au = 0$ in U. We assume that both X and A are real analytic.

4.1 Duality in Hardy spaces

Let $U \subset\subset X$ be a domain with real analytic boundary. In this and the next section we restrict ourselves to the case $\Sigma_1 = \mathcal{S}_\Delta(U)$ and $\Sigma_2 = \mathcal{S}_\Delta(\overline{U})$. Let $V = H_\Delta^{(B)}(U)$ and

$$
\begin{aligned}
i_1 : \quad H_\Delta^{(B)}(U) &\rightarrow \mathcal{S}_\Delta(U), \\
i_2 : \quad \mathcal{S}_\Delta(\overline{U}) &\rightarrow H_\Delta^{(B)}(U)
\end{aligned}
$$

be natural inclusions.

The mapping i_1 is always one-to-one, and the mapping i_2 is one-to-one because of the Unique Continuation Property $(U)_s$. As mentioned, the Poisson formula (2.24) implies the continuity of i_1. The mapping i_2 is continuous by a priori estimates for solutions of elliptic equations.

From the Runge theorem for solutions of elliptic systems (see for instance Theorem 4.1.26 in [Tar97]) it follows that $\mathcal{S}_\Delta(\overline{U})$ is dense in $\mathcal{S}_\Delta(U)$.

Our next goal is to extend the sesquilinear pairing $h(\cdot, \cdot)$, cf. (2.27), from $H_\Delta^{(B)}(U) \times H_\Delta^{(B)}(U)$ to $\Sigma_1 \times \Sigma_2$.

Note that the analyticity of ∂U implies that also $\partial D_{\pm\varepsilon}$ is real analytic for each sufficiently small $\varepsilon > 0$.

Theorem 4.1 *Let $0 \le j \le m - 1$ and let $\delta > 0$ be small enough. Then there exist a compact set $K \subset U$, $\varepsilon_0 > 0$ and $C > 0$ depending on j, δ, K and*

ε_0, *such that for all* $u \in \mathcal{S}_\Delta(U)$, $v \in \mathcal{S}_\Delta(U_\delta) \cap C^{2m}(\overline{U}_\delta, E)$, *and* $\varepsilon \in (0, \varepsilon_0]$ *we have*

$$\left| \int_{\partial U_{-\varepsilon}} (B_j u, B_j v)_x \, ds_{-\varepsilon} \right| \leq C \, \|u\|_{C(K,E)} \, \|v\|_{C^{2m}(\overline{U}_\delta, E)}. \qquad (4.1)$$

Proof. Let

$$n(f) \;=\; \bigoplus_{j=0}^{m-1} *_{F_j}^{-1} C_j *_F f$$

be the Dirichlet system adjoint to the system t with respect to the Green formula for A. Given any sufficiently small $\varepsilon > 0$, we consider the Cauchy problem

$$\begin{cases} A^* A v_{j,-\varepsilon} &= 0 & \text{near} & \partial U_{-\varepsilon}, \\ t(v_{j,-\varepsilon}) &= 0 & \text{on} & \partial U_{-\varepsilon}, \\ n(A v_{j,-\varepsilon}) &= (0,\ldots,0, B_j v, 0, \ldots, 0) & \text{on} & \partial U_{-\varepsilon} \end{cases} \qquad (4.2)$$

in a neighbourhood of $U_{-\varepsilon}$. The following statement is a consequence of the Cauchy-Kovalevskaya Theorem.

Lemma 4.2 *Given any* $\delta > 0$ *small enough, there are* $\varepsilon_0 > 0$ *and* $C_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0]$ *the sections* $v_{j,-\varepsilon}$ *belong to* $\mathcal{S}_\Delta(U_{-\varepsilon+r} \setminus \overline{U}_{-2\varepsilon_0})$ *with some* $r > 0$ *independent of* ε, *and satisfy*

$$\|v_{j,-\varepsilon}\|_{C^{2m}(\overline{U}_{-\varepsilon+r} \setminus U_{-2\varepsilon_0}, E)} \leq C_0 \, \|v\|_{C^{2m}(\overline{U}_\delta, E)}.$$

Proof. First we note that there exists a neighbourhood O of ∂U and sections $W_{j,-\varepsilon}$, real analytic near $\overline{O}$, such that

$$\begin{cases} t(W_{j,-\varepsilon}) &= 0 & \text{on} & \partial U_{-\varepsilon}, \\ n(A W_{j,-\varepsilon}) &= (0,\ldots,0, B_j v, 0, \ldots, 0) & \text{on} & \partial U_{-\varepsilon} \end{cases}$$

and

$$\|W_{j,-\varepsilon}\|_{C^{2m}(\overline{O}, E)} \leq C_0 \, \|v\|_{C^{2m}(\overline{U}_\delta, E)}. \qquad (4.3)$$

For instance, if $m_j = j$ we can take

$$W_{0,-\varepsilon}(x) \;=\; \frac{(\varrho(x) + \varepsilon)^{2m-1}}{(2m-1)!} \left(\sigma^{2m-1} (*_{F_0}^{-1} C_0 *_F A)(x, \nabla \varrho(x)) \right)^{-1} B_0 v(x),$$

$$W_{1,-\varepsilon}(x) \;=\; \frac{(\varrho(x) + \varepsilon)^{2m-2}}{(2m-2)!} \left(\sigma^{2m-2} (*_{F_1}^{-1} C_1 *_F A)(x, \nabla \varrho(x)) \right)^{-1} B_1 v(x) - \ldots$$

and so on.

Hence $v_{j,-\varepsilon} = W_{j,-\varepsilon} + w_{j,-\varepsilon}$ where $w_{j,-\varepsilon}$ is a solution of the following Cauchy problem

$$\begin{cases} A^* A w_{j,-\varepsilon} &= -A^* A W_{j,-\varepsilon} & \text{near} & \partial U_{-\varepsilon}, \\ t(w_{j,-\varepsilon}) &= 0 & \text{on} & \partial U_{-\varepsilon}, \\ n(A w_{j,-\varepsilon}) &= 0 & \text{on} & \partial U_{-\varepsilon}. \end{cases}$$

As the Dirichlet system $\{t, n \circ A\}$ represents the Cauchy data for the Laplacian Δ we see that

$$\begin{cases} A^* A w_{j,-\varepsilon} = -A^* A W_{j,-\varepsilon} & \text{near} \quad \partial U_{-\varepsilon}, \\ w_{j,-\varepsilon} \quad \text{vanishes up to order } 2m-1 & \text{on} \quad \partial U_{-\varepsilon}. \end{cases}$$

Since the hypersurfaces $\partial U_{-\varepsilon}$ are real analytic and compact we can argue locally. Fix a point $x^0 \in \partial U$. Then after a suitable bianalytic change of variables $x = \delta(y)$ in a neighbourhood of x^0 we obtain the following Cauchy problem

$$\begin{cases} \Delta^\sharp \, w_{j,-\varepsilon}(\delta(y)) = -A^* A W_{j,-\varepsilon}(\delta(y)) & \text{if} \quad y_n < -\varepsilon, \\ \left(\dfrac{\partial}{\partial y_n} \right)^k w_{j,-\varepsilon}(\delta(y)) \quad \text{vanishes for } j = 0, 1, \ldots, 2m-1 & \text{if} \quad y_n = -\varepsilon, \end{cases}$$
$$\tag{4.4}$$

where $\Delta^\sharp$ is a differential operator of order $2m$ with real analytic coefficients. Obviously, $\Delta^\sharp$ inherits the ellipticity from Δ.

Finally, complexifying problem (4.4) and using (4.3) and Theorem 9.4.5 from [Hoe83] we arrive at the assertion of the lemma.

$\square$

Further, let G_Δ be a Green operator for Δ. Then using Lemma 4.2 and Stokes' formula we get

$$\int_{\partial U_{-\varepsilon}} (B_j u, B_j v)_x \, ds_{-\varepsilon} = \int_{\partial U_{-\varepsilon}} (t(u), n(A v_{j,-\varepsilon}))_x \, ds_{-\varepsilon} - \int_{\partial U_{-\varepsilon}} (n(Au), t(v_{j,-\varepsilon}))_x \, ds_{-\varepsilon}$$

$$= \int_{\partial U_{-\varepsilon}} G_\Delta \left(*_E v_{j,-\varepsilon}, u \right)$$

$$= \int_{\partial U_{-\varepsilon_0}} G_\Delta \left(*_E v_{j,-\varepsilon}, u \right)$$

for all $0 < \varepsilon \le \varepsilon_0$.

Since $G_\Delta(\cdot, \cdot)$ is a bidifferential operator of order $2m-1$ on ∂U we conclude that

$$\left| \int_{\partial U_{-\varepsilon}} (B_j u, B_j v)_x \, ds_{-\varepsilon} \right| \le c \, \|u\|_{C^{2m-1}(\partial U_{-\varepsilon_0}, E)} \, \|v_{j,-\varepsilon}\|_{C^{2m-1}(\partial U_{-\varepsilon_0}, E)}, \tag{4.5}$$

the constant C depending on the coefficients of A and $\{B_j\}$ only. By Lemma 4.2, there is a constant $C_0 > 0$ such that

$$\|v_{j,-\varepsilon}\|_{C^{2m-1}(\partial U_{-\varepsilon_0}, E)} \le C_0 \, \|v\|_{C^{2m}(\overline{U}_\delta, E)}. \tag{4.6}$$

for all $v \in \mathcal{S}_\Delta(U_\delta) \cap C^{2m}(\overline{U}_\delta, E)$. Finally, by a priori estimates for solutions of elliptic systems there exists a constant $C_1 > 0$ such that for all $u \in \mathcal{S}_\Delta(U)$ we get

$$\|u\|_{C^{2m-1}(\partial U_{-\varepsilon_0}, E)} \le C_1 \, \|u\|_{C(\overline{U}_{-\varepsilon_0/2}, E)}. \tag{4.7}$$

Combining now inequalities (4.5), (4.6) and (4.7) we obtain (4.1), as desired.

$\square$

In the case where Δ is the usual Laplace operator in $\mathbb{R}^n$, Stout [Sto95] proved that ∂D should be necessarily real analytic for the estimate (4.1) to hold.

Corollary 4.3 *For every $u \in \mathcal{S}_\Delta(U)$ and $v \in \mathcal{S}_\Delta(\overline{U})$, the limit*

$$\tilde{h}(u,v) = \lim_{\varepsilon \to 0+} \int_{\partial D_{-\varepsilon}} (t(u), t(v))_x \, ds_{-\varepsilon}. \tag{4.8}$$

exists. The pairing $\tilde{h}(\cdot, \cdot)$ is separately continuous on $\mathcal{S}_\Delta(U) \times \mathcal{S}_\Delta(\overline{U})$, and $\tilde{h}(u,v) = h(u,v)$ for all $u \in H_\Delta^{(B)}(U)$ and $v \in \mathcal{S}_\Delta(\overline{U})$.

Proof. By the assumption, there exists a $\delta > 0$ with the property that $v \in \mathcal{S}_\Delta(U_\delta) \cap C^{2m}(\overline{U}_\delta, E)$.

Given any $\varepsilon \in (0, \varepsilon_0]$, ε_0 being from Theorem 4.1, we define a continuous functional $\mathcal{F}_{v,-\varepsilon}$ on $\mathcal{S}_\Delta(U)$ by

$$\mathcal{F}_{v,-\varepsilon}(u) = \int_{\partial U_{-\varepsilon}} (t(u), t(v))_x \, ds_{-\varepsilon},$$

for $u \in \mathcal{S}_\Delta(D)$. According to Theorem 4.1 there is a constant $C > 0$ independent of ε, such that

$$|\mathcal{F}_{v,-\varepsilon}(u)| \leq C \, \|u\|_{C(\overline{U}_{\varepsilon_0/2}, E)}$$

for all $0 < \varepsilon \leq \varepsilon_0$.

Let Σ be the subset of $\mathcal{S}_\Delta(U)$ consisting of all solutions u with the property that

$$\|u\|_{C(\overline{U}_{\varepsilon_0/2}, E)} < \frac{1}{C}.$$

This means that for each $\varepsilon \in (0, \varepsilon_0]$ the functional $\mathcal{F}_{v,-\varepsilon}$ belongs to the polar of Σ, i.e.,

$$\Sigma^\circ = \{ \mathcal{F} \in \mathcal{S}_\Delta(U)' : \quad |\mathcal{F}(u)| \leq 1 \text{ for all } u \in \Sigma \}.$$

By a familiar theorem of Alaoglu and Banach, this polar is weak* compact. Since the space $\mathcal{S}_\Delta(U)$ is separable, this polar is metrisable in the weak* topology. By compactness, there are limit points for the net $\{\mathcal{F}_{v,-\varepsilon}\}_{0 < \varepsilon \leq \varepsilon_0}$. Let $\mathcal{F}_0$ be such a limit point. Thus, for some sequence $\varepsilon_k \in (0, \varepsilon_0]$ converging to 0, we have

$$\lim_{k \to \infty} \mathcal{F}_{v,-\varepsilon_k} = \mathcal{F}_0(u)$$

for all $u \in \mathcal{S}_\Delta(U)$.

It is easy to see that for any $u \in H_\Delta^{(B)}(U)$ we have

$$
\begin{aligned}
\mathcal{F}_0(u) &= \lim_{k \to \infty} \mathcal{F}_{v,-\varepsilon_k}(u) \\
&= h(u, v).
\end{aligned}
$$

Hence, each weak* limit point of the net $\{\mathcal{F}_{v,-\varepsilon}\}_{0 < \varepsilon \le \varepsilon_0}$ agrees on $H_\Delta^{(B)}(U)$ with $u \mapsto h(u, v)$.

As $\mathcal{S}_\Delta(\overline{U})$ is dense in $\mathcal{S}_\Delta(U)$, the space $H_\Delta^{(B)}(U)$ is dense there, too. This implies the existence of a limit

$$
\lim_{\varepsilon \to 0+} \mathcal{F}_{v,-\varepsilon} = \mathcal{F}_0
$$

which defines an element of $\mathcal{S}_\Delta(U)'$.

Finally, the separate continuity of the pairing $\tilde{h}(\cdot, \cdot)$ follows immediately from (4.1).

□

Theorem 4.4 *The mapping* $\mathcal{J} : \mathcal{S}_\Delta(\overline{U}) \to \mathcal{S}_\Delta(U)'$ *induced by (4.8) is a topological isomorphism of these spaces.*

Proof. Since the natural inclusion i_2 is one-to-one, the mapping $\mathcal{J}$ is one-to-one, too (see Lemma 2.2).

According to Corollary 2.5 to prove the surjectivity of the mapping we have to show that the reproducing kernel $K(\cdot, \cdot)$ of the space $H_\Delta^{(B)}(U)$ has the following property.

Lemma 4.5 *For every fixed* $x \in U$, *the Szegö kernel* $K(x, \cdot)$ *belongs to* $E_x^* \otimes \mathcal{S}_\Delta(\overline{U})$.

Proof. Since X, ∂D, Δ and $\{B_j\}$ are real analytic Theorem A of [MN57] implies that any solution u of the Dirichlet problem (2.20) actually satisfies $\Delta u = 0$ in a neighbourhood of $\overline{U}$ if the data $\oplus_{j=0}^{m-1} u_j$ are real analytic, cf. for instance Lemma 4.4 in [NST98]. This means, in particular, that $G_U(x, \cdot)$ is real analytic in a neighbourhood of ∂U. Hence, we deduce from (2.29) that $K(x, \cdot) \in E_x \otimes \mathcal{S}_\Delta(\overline{U})$ for every fixed $x \in U$, as desired.

□

We have thus proved that the mapping $\mathcal{J}$ is an isomorphism of vector spaces

$$
\mathcal{S}_\Delta(\overline{U}) \overset{\cong}{\hookrightarrow} \mathcal{S}_\Delta(U)'.
$$

We are now going to invoke an operator-theoretic argument to conclude that this algebraic isomorphism is in fact a topological one. To this end, we note that the spaces $\mathcal{S}_\Delta(\overline{U})$ and $\mathcal{S}_\Delta(U)'$ are both spaces of type *DFS*. For

$\mathcal{S}_\Delta(\overline{U})$, see the proof of Theorem 1.5.5 in Morimoto [Mor93, p.13]. As the Closed Graph Theorem is correct for linear mappings between spaces of type *DFS* (see Corollary A.6.4 in [Mor93, p.254]), to see that $v \mapsto \mathcal{F}_v$ is a topological isomorphism, it suffices to show that it is continuous. The latter conclusion, however, follows from Theorem 4.1 and the explicit construction of $\mathcal{F}_v$. This completes the proof.

$\square$

Recently Stout [Sto95] proved Theorem 4.4 for the usual Laplace operator Δ in $\mathbb{R}^n$.

As we have a commutative diagram

$$
\begin{array}{ccccc}
\mathcal{S}_\Delta(\overline{U}) & \overset{i_2}{\hookrightarrow} & H_\Delta^{(B)}(U) & \overset{i_1}{\hookrightarrow} & \mathcal{S}_\Delta(U) \\
\cup & & \cup & & \cup \\
\mathcal{S}_A(\overline{U}) & \overset{i_2}{\hookrightarrow} & H_A^{(B)}(U) & \overset{i_1}{\hookrightarrow} & \mathcal{S}_A(U)
\end{array}
$$

the pairing $\tilde{h}(\cdot, \cdot)$ induces a continuous mapping

$$
\mathfrak{J}: \ \mathcal{S}_A(\overline{U}) \to \mathcal{S}_A(U)'
$$

which is the restriction of $\mathcal{J}$.

Write π for the Szegö projection

$$
\pi: \ H_\Delta^{(B)}(U) \to H_A^{(B)}(U).
$$

Corollary 4.6 *The mapping $\mathfrak{J}$ is a topological isomorphism of the space $\mathcal{S}_A(\overline{U})$ onto $\mathcal{S}_A(U)'$ if and only if*

1) $\mathcal{S}_A(\overline{U})$ *is dense in* $\mathcal{S}_A(U)$;

2) π *maps* $\mathcal{S}_\Delta(\overline{U})$ *continuously into* $\mathcal{S}_A(\overline{U})$.

Proof. According to the general scheme we have

$$
\begin{array}{rcl}
\mathfrak{S}_1 & = & \mathcal{S}_A(U), \\
\mathfrak{S}_2 & = & \mathcal{S}_A(\overline{U}), \\
\mathfrak{V} & = & H_A^{(B)}(U),
\end{array}
$$

hence the statement follows from Corollary 2.7.

$\square$

Stout [Sto95] proved this theorem for the overdetermined Cauchy-Riemann operator $A = \overline{\partial}$ in $\mathbb{C}^n$, $n > 1$.

In [Shl00] it is proved that conditions 1) and 2) of Corollary 4.6 hold for a Dirac operator A in a ball of $X = \mathbb{R}^n$, cf. Example 3.5.

Example 4.7 Let A be a determined elliptic operator with real analytic coefficients on X. Then both A and A^* possess the Unique Continuation Property in the interior of X. Again condition 1) of Corollary 4.6 is fulfilled.

As in Example 3.6 we consider the extension of A to an operator

$$T : \ H_\Delta^{(B)}(U) \to H_{A^*}^{(C)}(U)$$

where $H_{A^*}^{(C)}(U)$ is the closed subspace of the Hardy space $H_{AA^*}^{(C)}(U)$ consisting of all solutions to $A^* g = 0$ in U.

As is proved in Example 3.6, for any $g \in H_{A^*}^{(C)}(U)$ we have

$$(A P_U \, n)^{-1} \, g = P_U^1 t(\Phi_U g)$$

where P_U^1 the Poisson integral of the Dirichlet problem for AA^* in U.

Let $u \in \mathcal{S}_\Delta(\overline{U})$. Then $Au \in \mathcal{S}_{A^*}(\overline{U})$ and hence it is real analytic in a neighbourhood of $\overline{U}$. By Stokes' formula,

$$\Phi_U (Au) \, (x) = u(x) + \int_{\partial U} G_A(K_\Phi(x, \cdot), u)$$

for all $x \in U$.

If $u \in \mathcal{S}_\Delta(U_\delta) \cap C^{2m}(\overline{U}_\delta, E)$ with $\delta > 0$ sufficiently small, then by the Cauchy-Kovalevskaya Theorem there exist a neighbourhood O of ∂U and a section $v \in C^m(\overline{O}, E)$ such that

$$\begin{cases} Av & = & 0 & \text{in} & O, \\ t(v) & = & t(u) & \text{on} & \partial U. \end{cases}$$

By Stokes' formula,

$$\Phi_U (Au) \, (x) = u(x) - v(x) + \int_{\partial O \cap U} G_A(K_\Phi(x, \cdot), v) \qquad (4.9)$$

for all $x \in O \cap U$.

By the definition of v, the right-hand side of the last equality uniquely extends to a solution of $Au = 0$ in O. It follows that $\Phi_U(Au) \in \mathcal{S}_\Delta(O \cup U)$.

Arguing as in the proof of Lemma 4.2 we see that the neighbourhood O does not depend on u but does on δ and ∂U. Moreover, for every solution $u \in \mathcal{S}_\Delta(U_\delta) \cap C^m(\overline{U}_\delta, E)$ we get

$$\|v\|_{C^m(\overline{O}, E)} \le C \, \|u\|_{C^m(\overline{U}_\delta, E)}$$

with $C > 0$ a constant independent of u. Hence $\Phi_U(Au)$ maps $\mathcal{S}_\Delta(\overline{U})$ continuously into $\mathcal{S}_\Delta(\overline{U})$.

Further, since X, ∂U, AA^* and $\{B_j\}$ are real analytic, Theorem A of [MN57] implies that the Poisson integral $P_U^1 \Phi_U(Au)$ is real analytic in a neighbourhood of ∂U, provided that $u \in \mathcal{S}_\Delta(\overline{U})$. Therefore, πu is real analytic in a neighbourhood of ∂U, too.

If $\{u_\nu$ converges to $u \in \mathcal{S}_\Delta(\overline{U})$ then $\Phi_U(Au_\nu)$ converges to $\Phi_U u$ by (4.9), and $P_U^1 \Phi_U(Au_\nu)$ converges to $P_U^1 \Phi_U(Au)$ by [NST98] (see the proof of Lemma 4.4). Hence π maps $\mathcal{S}_\Delta(\overline{U})$ continuously into $\mathcal{S}_A(\overline{U})$, cf. Section 2.3.

Summarising we conclude that the mapping $\mathfrak{J}$ is a topological isomorphism of $\mathcal{S}_A(\overline{U})$ onto $\mathcal{S}_A(U)'$.

Example 4.8 Suppose A is an overdetermined elliptic differential operator with constant coefficients in $\mathbb{R}^n$, as in Example 3.7. If $U \subset\subset \mathbb{R}^n$ is a strictly convex domain with real analytic boundary then under reasonable assumptions on A the Neumann problem for the tangential complex is solvable at step 1, see Example 3.7 for more details.

Moreover, the Neumann operator N^1 possesses the analytic hypoellipticity property, cf. [Tar95a]. Hence the operator $I - T_b^* N T_b$, i.e., the orthogonal projection from $L^2(\partial U, E_b^0)$ to $\ker T_b^0$, maps $\mathcal{S}_\Delta(\overline{U})$ continuously to $\mathcal{S}_A(\overline{U})$. As $L^2(\partial U, E_b^0) = H_\Delta^{(B)}(U)$ and $\ker T_b^0 = H_A^{(B)}(U)$ we see that the projection π has the same property.

As the domain U is strictly convex we see that $\mathcal{S}_A(\overline{U})$ is dense in $\mathcal{S}_A(U)$. By Corollary 3.11, the mapping $\mathfrak{J}$ is a topological isomorphism of $\mathcal{S}_A(\overline{U})$ onto $\mathcal{S}_A(U)'$. By reflexivity, the transpose $\mathfrak{J}'$ gives us a topological isomorphism between the spaces $\mathcal{S}_A(\overline{U})'$ and $\mathcal{S}_A(U)$.

4.2   Duality in Lebesgue spaces

We will now study the case

$$\begin{aligned}
\Sigma_1 &= \mathcal{S}_\Delta(U), \\
\Sigma_2 &= \mathcal{S}_\Delta(\overline{U}), \\
V &= \mathcal{S}_\Delta^{(0)}(U)
\end{aligned}$$

and

$$\begin{aligned}
i_1 &: \mathcal{S}_\Delta^{(0)}(U) \rightarrow \mathcal{S}_\Delta(U), \\
i_2 &: \mathcal{S}_\Delta(\overline{U}) \rightarrow \mathcal{S}_\Delta^{(0)}(U)
\end{aligned}$$

are natural inclusions.

The mapping i_1 is always one-to-one and the mapping i_2 is one-to-one because of the Unique Continuation Property $(U)_s$. By a priori elliptic estimates, the mappings i_1 and i_2 are continuous. As mentioned, $\mathcal{S}_\Delta^{(0)}(U)$ is a separable Hilbert space with reproducing kernel. To proceed we thus need to extend the

pairing

$$h(u, v) = \int_U (u, v)_x \, dx$$

from $L^2(U, E) \times L^2(U, E)$ to $\Sigma_1 \times \Sigma_2$.

Corollary 4.9 *Let $\delta > 0$ be small enough. Then there are a compact set $K \subset U$, $\varepsilon_0 > 0$ and $C > 0$ depending on δ, K and ε_0, such that for all $u \in \mathcal{S}_\Delta(U)$ and $v \in \mathcal{S}_\Delta(U_\delta) \cap C^{2m}(\overline{U}_\delta, E)$ we have*

$$\left| \int_{U_{-\varepsilon}} (u, v)_x \, dx \right| \le C \, \|u\|_{C(K,E)} \, \|v\|_{C^{2m}(\overline{U}_\delta, E)} \tag{4.10}$$

whenever $\varepsilon \in (0, \varepsilon_0]$.

Proof. Let ε_0 be the number from Theorem 4.1. Since ∂U is sufficiently smooth there exists $0 < \varepsilon' \le \varepsilon_0$ such that for $0 < \varepsilon \le \varepsilon'$ we have

$$\int_{U_{-\varepsilon}} (u, v)_x \, dx = \int_{U_{-\varepsilon'}} (u, v)_x \, dx + \int_\varepsilon^{\varepsilon'} dr \int_{\partial U_{-r}} (u, v)_x \, ds_{-r},$$

whence

$$\left| \int_{U_{-\varepsilon}} (u, v)_x \, dx \right| \le \text{meas}(U) \, \|u\|_{C(\overline{U}_{-\varepsilon'}, E)} \|v\|_{C(\overline{U}_{-\varepsilon'}, E)} + \varepsilon' \sup_{r \in [\varepsilon, \varepsilon']} \left| \int_{\partial U_{-r}} (u, v)_x \, ds_{-r} \right|.$$

Now the statement of the corollary follows from Theorem 4.1 with $B_0 = I$, as desired.

$\square$

In case Δ is the usual Laplace operator in $\mathbb{R}^n$, Zorn [Zor82] proved that the boundary of U should be necessarily real analytic in order that the estimate (4.10) may hold.

Corollary 4.10 *For every solutions $u \in \mathcal{S}_\Delta(U)$ and $v \in \mathcal{S}_\Delta(\overline{U})$ there exists a limit*

$$\tilde{h}(u, v) = \lim_{\varepsilon \to 0+} \int_{U_{-\varepsilon}} (u, v)_x \, dx. \tag{4.11}$$

The corresponding pairing $\tilde{h}(\cdot, \cdot)$ is separately continuous on $\mathcal{S}_\Delta(U) \times \mathcal{S}_\Delta(\overline{U})$, and $\tilde{h}(u, v) = h(u, v)$ for all $u \in \mathcal{S}_\Delta^{(0)}(U)$ and $v \in \mathcal{S}_\Delta(\overline{U})$.

Proof. The proof is similar to the proof of Corollary 4.3.

$\square$

Theorem 4.11 *The mapping $\mathcal{J} : \mathcal{S}_\Delta(\overline{U}) \to \mathcal{S}_\Delta(U)'$ induced by (4.11) is a topological isomorphism of these spaces.*

Proof. Since the natural inclusion i_2 is one-to-one, the mapping $\mathcal{J}$ is one-to-one, too (see Lemma 2.2).

By Corollary 2.5, to prove the surjectivity of the mapping we have to show that the reproducing kernel $K(\cdot,\cdot)$ of the space $\mathcal{S}_\Delta^{(0)}(U)$ has the following property.

Lemma 4.12 *For every fixed $x \in U$, the Bergman kernel $K(x,\cdot)$ belongs to $E_x^* \otimes \mathcal{S}_\Delta(\overline{U})$.*

Proof. In the proof of Lemma 3.9 we derived the formula

$$\Delta V(x,\cdot) = -K(x,\cdot)$$

in U, for any fixed $x \in U$, where $V(x,\cdot)$ is a solution of the Dirichlet problem (3.6) for the operator Δ^2. Since all the objects X, ∂U, Δ and $\{B_j\}$ are real analytic we deduce by Theorem A of [MN57] that $V(x,\cdot) \in E_x^* \otimes \mathcal{S}_{\Delta^2}(\overline{U})$ for every fixed $x \in U$ (see for instance Lemma 4.4 in [NST98]). Hence the lemma follows, as desired.

$\square$

We have proved that the mapping $\mathcal{J}$ is an isomorphism of vector spaces $\mathcal{S}_\Delta(\overline{U})$ and $\mathcal{S}_\Delta(U)'$. The topological arguments are actually the same as those in the proof of Theorem 4.4.

$\square$

Since we have a commutative diagram

$$
\begin{array}{ccccc}
\mathcal{S}_\Delta(\overline{U}) & \overset{i_2}{\hookrightarrow} & \mathcal{S}_\Delta^{(0)}(U) & \overset{i_1}{\hookrightarrow} & \mathcal{S}_\Delta(U) \\
\cup & & \cup & & \cup \\
\mathcal{S}_A(\overline{U}) & \overset{i_2}{\hookrightarrow} & \mathcal{S}_A^{(0)}(U) & \overset{i_1}{\hookrightarrow} & \mathcal{S}_A(U)
\end{array}
$$

the pairing $\tilde{h}(\cdot,\cdot)$ induces a continuous mapping

$$\mathfrak{J}: \ \mathcal{S}_A(\overline{U}) \to \mathcal{S}_A(U)'$$

which is the restriction of $\mathcal{J}$.

Write π for the Bergman projection

$$\pi: \ \mathcal{S}_\Delta^{(0)}(U) \to \mathcal{S}_A^{(0)}(U).$$

Corollary 4.13 *In order that the mapping $\mathfrak{J}$ be a topological isomorphism of the space $\mathcal{S}_A(\overline{U})$ onto $\mathcal{S}_A(U)'$ it is necessary and sufficient that*

1) *$\mathcal{S}_A(\overline{U})$ be dense in $\mathcal{S}_A(U)$;*
2) *π would map $\mathcal{S}_\Delta(\overline{U})$ continuously into $\mathcal{S}_A(\overline{U})$.*

Proof. According to the general scheme we have

$$
\begin{aligned}
\mathfrak{S}_1 &= \mathcal{S}_A(U), \\
\mathfrak{S}_2 &= \mathcal{S}_A(\overline{U}), \\
\mathfrak{V} &= \mathcal{S}_A^{(0)}(U),
\end{aligned}
$$

and so the statement follows from Corollary 2.7.

$\square$

Zorn [Zor82] proved that the conditions of Corollary 4.13 hold for the Cauchy-Riemann operator $A = \bar{\partial}$ in any strictly pseudoconvex domain U of $\mathbb{C}^n$ with real analytic boundary.

Example 4.14 Let A be a determined elliptic operator as in Example 3.12. Since X, ∂U, AA^* and $n(\cdot)$ are real analytic Theorem A of [MN57] implies that NAu is real analytic in a neighbourhood of ∂U if $u \in \mathcal{S}_\Delta(\overline{U})$. Therefore $\pi u = u - A^* N A u$ is real analytic in a neighbourhood of ∂U, too. Furthermore, if $\{u_\nu\}$ converges to $u \in \mathcal{S}_\Delta(\overline{U})$ then Nu_ν converges to Nu, cf. the proof of Lemma 4.4 in [NST98]. Hence π maps $\mathcal{S}_\Delta(\overline{U})$ continuously onto $\mathcal{S}_A(\overline{U})$. Finally, by the Runge theorem for determined elliptic operators, cf. [Tar97, 4.1.9], $\mathcal{S}_A(\overline{U})$ is dense in $\mathcal{S}_A(U)$. Hence, according to Corollary 4.13, the mapping $\mathfrak{J}$ is a topological isomorphism of $\mathcal{S}_A(\overline{U})$ onto $\mathcal{S}_A(U)'$. By reflexivity, the transposed mapping $\mathfrak{J}'$ is a topological isomorphism of $\mathcal{S}_A(\overline{U})'$ onto $\mathcal{S}_A(U)$.

Example 4.15 Assume that A is a column of first order scalar partial differential operators with constant coefficients in $\mathbb{R}^n$. Under familiar assumptions on A, the compatibility complex of A is simply a Koszul complex, cf. [Tar95a, 1.2.8]. Let $U \subset\subset \mathbb{R}^n$ be a strictly convex domain with real analytic boundary. Then the Neumann problem for the compatibility complex in U is solvable at step 1, and the Neumann operator N preserves real analytic sections in a neighbourhood of $\overline{U}$. The latter remains still valid with "convex" replaced by "pseudoconvex" in an appropriate sense, cf. [Tar95a, 4.1.5]. As the domain U is strictly convex, the subspace $\mathcal{S}_A(\overline{U})$ is dense in $\mathcal{S}_A(D)$. By Corollary 4.13, the mapping $\mathfrak{J}$ arranges a topological isomorphism of $\mathcal{S}_A(\overline{U})$ onto $\mathcal{S}_A(U)'$.

Example 4.16 Let $X = \mathbb{R}^n$, $n \geq 3$, $U \subset \mathbb{R}^n$ a ball, and A a Dirac operator in $\mathbb{R}^n$, cf. Example 3.5. It is proved in [Shl96] that there are systems $\{b_\nu\}$ and $\{c_\mu\}$ of ($\mathbb{C}^k$-valued) homogeneous harmonic polynomials, such that

a) $\{b_\nu\}$ is an orthogonal basis in all spaces $\mathcal{S}_A^{(0)}(U_\varepsilon)$, $\varepsilon \in \mathbb{R}$, simultaneously;

b) $\{b_\nu\} \cup \{c_\mu\}$ is an orthogonal basis in all spaces $\mathcal{S}_\Delta^{(0)}(U_\varepsilon)$, $\varepsilon \in \mathbb{R}$, simultaneously.

Property a) implies that condition 1) of Corollary 3.11 is satisfied. Moreover, the projection π restricted to $\mathcal{S}_\Delta^{(0)}(U_\varepsilon)$, $\varepsilon > 0$, coincides with the orthogonal projection

$$\mathcal{S}_\Delta^{(0)}(U_\varepsilon) \to \mathcal{S}_A^{(0)}(U_\varepsilon).$$

It follows that π maps $\mathcal{S}_\Delta(\overline{U})$ continuously into $\mathcal{S}_A(\overline{U})$. Hence Corollary 3.20 holds for A.

4.3   Grothendieck duality

Suppose U is an open subset of $\overset{\circ}{X}$ with C^∞ boundary, such that $\partial X \cap \overline{U} = \emptyset$ or $\partial X \cap \overline{U} = \partial X$. Set

$$
\begin{aligned}
\Sigma_1 &= \frac{\mathcal{S}_\Delta(U)}{\mathcal{H}(X)}, \\
\Sigma_2 &= \frac{\{u \in \mathcal{S}_\Delta(X \setminus U) : \ t(u) = 0 \text{ on } \partial X\}}{\mathcal{H}(X)}
\end{aligned}
$$

and

$$V = \mathcal{S}_\Delta^{(m)}(X \setminus \overline{U}, \partial X) \ominus \mathcal{H}(X),$$

"$\ominus$" meaning the orthogonal complement with respect to the scalar product $h_{X\setminus\overline{U}}(\cdot,\cdot)$.

For each $[u] \in \Sigma_2$, we set

$$i_2[v] = p(v)$$

where v is a representative of the class $[v]$ and $p(v)$ the orthogonal projection of v to V in $\mathcal{S}_\Delta^{(m)}(X \setminus \overline{U}, \partial X)$. If $v_1, v_2 \in [v]$ then $v_1 - v_2 \in \mathcal{H}(X)$ whence $p(v_1 - v_2) = 0$. It follows that the mapping $i_2 : \Sigma_2 \to V$ is well defined and continuous, as is easy to check.

Further, for $u \in V$ we set

$$i_1 u = \left[\mathcal{E}_{X\setminus\overline{U}}(u)\right],$$

i.e., the equivalence class in Σ_1 corresponding to the restriction of $\mathcal{E}_{X\setminus\overline{U}}(u)$ to U.

We are now in a position to extend the sesquilinear pairing $h_{X\setminus\overline{U}}(\cdot,\cdot)$ from $V \times V$ to $\Sigma_1 \times \Sigma_2$. Namely, if $[v] \in \Sigma_2$ then there exists a domain $O \subset\subset U$ with smooth boundary ∂O, such that $v \in \mathcal{S}_\Delta(X \setminus O)$ for all $v \in [v]$. Given any $[u] \in \Sigma_1$ and $[v] \in \Sigma_2$, we set

$$\tilde{h}_{X\setminus\overline{U}}([u],[v]) = -\int_{\partial O} G_\Delta \left(*_E v, u\right) \tag{4.12}$$

where $u \in [u]$ and $v \in [v]$.

Lemma 4.17 *As defined by (4.12), the pairing $\tilde{h}_{X\backslash\overline{U}}(\cdot,\cdot)$ does not depend on the choice of O and $u \in [u]$, $v \in [v]$. Moreover, it is separately continuous, and*

$$\tilde{h}_{X\backslash\overline{U}}(i_1 u, [v]) = h_{X\backslash\overline{U}}(u, i_2[v])$$

for all $u \in V$ and $[v] \in \Sigma_2$.

Proof. By Stokes' formula, we get

$$\int_{\partial O} G_\Delta (*_E v, u) = \int_O (\Delta u, v)_x \, dx - \int_O (u, \Delta v)_x \, dx$$
$$= 0$$

for all $v \in \mathcal{H}(X)$. Similarly,

$$-\int_{\partial O} G_\Delta (*_E v, u) = \int_{X\backslash O} (\Delta u, v)_x \, dx - \int_{X\backslash O} (u, \Delta v)_x \, dx$$
$$= 0$$

for all $u \in \mathcal{H}(X)$. This means that $\tilde{h}_{X\backslash\overline{U}}(\cdot,\cdot)$ does not depend on the choice of $u \in [u]$ and $v \in [v]$.

Let $O' \subset\subset U$ be another domain with smooth boundary, such that $\Delta v = 0$ near $X \backslash O'$ for all $v \in [v]$. Without loss of generality we can assume that $O \subset\subset O'$. Then by Stokes' formula we get

$$\int_{\partial O'} G_\Delta (*_E v, u) - \int_{\partial O} G_\Delta (*_E v, u) = \int_{\partial(O'\backslash O)} G_\Delta (*_E v, u)$$
$$= 0, \qquad (4.13)$$

for both u and v belong to $\mathcal{S}_\Delta(\overline{O}' \backslash O)$. Thus, $\tilde{h}_{X\backslash\overline{U}}(\cdot,\cdot)$ is independent of the particular choice of O.

Obviously, $\tilde{h}_{X\backslash\overline{U}}(\cdot,\cdot)$ is separately continuous if the spaces Σ_1 and Σ_2 are endowed with canonical quotient topology.

Since any solution $u \in V$ has finite order of growth near ∂U it follows from (4.13) that

$$\tilde{h}_{X\backslash\overline{U}}(i_1 u, [v]) = -\int_{\partial O} G_\Delta \left(*_E v, \mathcal{E}_{X\backslash\overline{U}} u\right)$$
$$= -\int_{\partial U} G_\Delta \left(*_E v, \mathcal{E}_{X\backslash\overline{U}} u\right)$$
$$= -\int_{\partial U} \left((t(\mathcal{E}_{X\backslash\overline{U}} u), n(Av))_x - (n(A\mathcal{E}_{X\backslash\overline{U}} u), t(v))_x\right) ds$$
$$= \int_{\partial U} \left((n(A\mathcal{E}_{X\backslash\overline{U}} u), t(\mathcal{E}_{X\backslash\overline{U}} v))_x - (t(u), n(Av))_x\right) ds$$
$$= h_{X\backslash\overline{U}}(u, v) - \int_X (H\mathcal{E}_{X\backslash\overline{U}} u, H\mathcal{E}_{X\backslash\overline{U}} v)_x dx$$
$$= h_{X\backslash\overline{U}}(u, v),$$

the last equality being a consequence of the fact that $u \perp \mathcal{H}(X)$ with respect to $h_{X \setminus \overline{U}}(\cdot, \cdot)$.

Finally, we obtain

$$
\begin{aligned}
h_{X \setminus \overline{U}}(u, v) &= h_{X \setminus \overline{U}}(p(u), v) \\
&= h_{X \setminus \overline{U}}(u, p(v)) \\
&= h_{X \setminus \overline{U}}(u, i_2[v]),
\end{aligned}
$$

showing the lemma.

$\square$

We thus conclude that there is a mapping $\mathcal{J} : \Sigma_2 \to \Sigma_1'$ induced by the pairing $\tilde{h}_{X \setminus \overline{U}}(\cdot, \cdot)$.

Theorem 4.18 *The mapping*

$$
\mathcal{J} : \frac{\{u \in \mathcal{S}_\Delta(X \setminus U) : \ t(u) = 0 \ on \ \partial X\}}{\mathcal{H}(X)} \to \left(\frac{\mathcal{S}_\Delta(U)}{\mathcal{H}(X)}\right)'
$$

induced by (4.12) is a topological isomorphism of these spaces.

Proof. As $p(u) = 0$ implies $u \in \mathcal{H}(X)$ we see that i_2 is one-to-one. Then $\mathcal{J}$ is one-to-one, too (see Lemma 2.2).

Let us prove the surjectivity of $\mathcal{J}$. To this end we pick a continuous linear functional $\mathcal{F}$ on Σ_1. Then $\mathcal{F}$ can be thought of as a functional on $\mathcal{S}_\Delta(U)$ vanishing on $\mathcal{H}(X)$.

Since $\mathcal{S}_\Delta(U)$ is a subspace of $C_{\mathrm{loc}}(U, E)$, the space of continuous sections of E over U, this functional can be extended, by the Hahn-Banach Theorem, to an E^*-valued measure m with compact support in U orthogonal to $\mathcal{H}(X)$.

Take a domain $O' \subset\subset U$ containing the support of m. Then for every $u \in \mathcal{S}_\Delta(U)$ and $x \in O'$ we have

$$
u(x) = - \int_{\partial O'} G_\Delta(K_G(x, \cdot), u) + H(\chi_{O'} u)(x)
$$

where G is the Hodge parametrix for the Dirichlet problem in X, cf. Section 2.4.

Hence

$$
\langle \mathcal{F}, [u] \rangle = - \int_{\partial O'} G_\Delta(*_E v, u) \tag{4.14}
$$

with an element $u \in [u]$, and

$$
\begin{aligned}
v(y) &= *_E^{-1} \langle dm, K_G(\cdot, y) \rangle \\
&= G\left(*_E^{-1} dm\right)(y)
\end{aligned}
$$

for y away from the support of m.

Since G is a Hodge parametrix we see that $t(v) = 0$ on ∂X. Moreover, we get

$$
\begin{aligned}
\Delta v &= *_E^{-1} dm - H\left(*_E^{-1} \mathcal{F} \right) \\
&= *_E^{-1} dm,
\end{aligned}
$$

for $\mathcal{F}$ vanishes on $\mathcal{H}(X)$. It follows that $\Delta v = 0$ in a neighbourhood of $X \setminus O'$, i.e., v determines an equivalence class $[v]$ in Σ_2. Obviously, we have $\mathcal{J}[v] = \mathcal{F}$, which proves the surjectivity of $\mathcal{J}$.

Finally, the topological arguments are actually the same as those in Theorem 4.4.

$\square$

4.4 Duality in Sobolev spaces

From now on we will assume that the manifold X, the bundles E, F and the coefficients of the operator A are real analytic.

Let $U \subset\subset X$ be a domain with real analytic boundary. We introduce the spaces of solutions

$$
\begin{aligned}
\Sigma_1 &= \frac{\mathcal{S}_\Delta(U)}{\mathcal{H}(X)}, \\
\Sigma_2 &= \frac{\mathcal{S}_\Delta(\overline{U})}{\mathcal{H}(X)}
\end{aligned}
$$

and

$$
V = \mathcal{S}_\Delta^{(m)}(U) \ominus \mathcal{H}(X),
$$

'$\ominus$' meaning the orthogonal complement in $\mathcal{S}_\Delta^{(m)}(U)$ with respect to the scalar product $h_U(\cdot, \cdot)$. We endow V with the scalar product $h_U(\cdot, \cdot)$, thus making it a Hilbert space.

For $[u] \in \Sigma_2$, we set

$$
E_U[v] = [\mathcal{E}_U(u)]
$$

with u a representative of $[u]$. Let us check that this definition does not depend on the particular choice of $u \in [u]$. Indeed, it was proved in [NST98] (see the proof of Corollary 4.1) that the mapping

$$
\mathcal{E}_U : \ \mathcal{S}_\Delta(\overline{U}) \to \{ u \in \mathcal{S}_\Delta(X \setminus U) : \ \ t(u) = 0 \text{ on } \partial X \ \}
$$

is a topological isomorphism of the spaces. Therefore, we conclude that $\mathcal{E}_U(u)$ belongs to $\mathcal{S}_\Delta(X \setminus U)$ and satisfies $t(\mathcal{E}_U(u)) = 0$ on ∂X. In particular, if $u \in \mathcal{H}(X)$ then $\mathcal{E}_U(u) = u$ and this gives us the independence on the choice of $u \in [u]$, as desired.

As the space $\mathcal{H}(X)$ is finite dimensional we immediately obtain the following lemma.

Lemma 4.19 *The mapping*

$$E_U : \; \frac{\mathcal{S}_\Delta(\overline{U})}{\mathcal{H}(X)} \to \frac{\{u \in \mathcal{S}_\Delta(X \setminus U) : \;\; t(u) = 0 \;\; on \;\; \partial X \; \}}{\mathcal{H}(X)}$$

is a topological isomorphism of the spaces.

Of course, Lemma 4.19 and Theorem 4.18 already imply that the spaces Σ_2 and Σ_1' are topologically isomorphic. However, we want to derive an explicit construction of this duality.

To this end, we set

$$\begin{array}{rcll} i_1 & : & V & \to & \Sigma_1, \\ i_2 & : & \Sigma_2 & \to & V \end{array}$$

to be the natural inclusions. They are obviously one-to-one and continuous.

We define an extension $\tilde{h}_U(\cdot,\cdot)$ of $h_U(\cdot,\cdot)$ as follows:

$$\tilde{h}_U([u],[v]) = \tilde{h}_{X \setminus \overline{U}}\left([u], E_U[v]\right), \tag{4.15}$$

cf. (3.11).

Lemma 4.20 *As defined by (4.15), the pairing $\tilde{h}_U(\cdot,\cdot)$ does not depend on the choice of $u \in [u]$ and $v \in [v]$. Moreover, it is separately continuous and satisfies*

$$\tilde{h}_U\left(i_1 u, [v]\right) = h_U\left(u, i_2[v]\right)$$

for all $u \in V$ and $[v] \in \Sigma_2$.

Proof. The pairing is independent of the choice of $u \in [u]$ and $v \in [v]$ because so are the pairing $\tilde{h}_{X \setminus \overline{U}}$ and the mapping E_U.

Moreover, from the definition of $\mathcal{E}_U$ it follows that $\mathcal{E}_{X \setminus \overline{U}} \mathcal{E}_U = 1$ on $\mathcal{S}_\Delta^{(m)}(U)$. Hence, by Lemma 4.17 we get

$$\begin{aligned} \tilde{h}_U\left(i_1 u, [v]\right) & = & \tilde{h}_{X \setminus \overline{U}}\left(i_1 \mathcal{E}_{X \setminus \overline{U}} \mathcal{E}_U(u), E_U([v])\right) \\ & = & h_{X \setminus \overline{U}}\left(\mathcal{E}_{X \setminus \overline{U}} \mathcal{E}_U(u), i_2 E_U([v])\right) \\ & = & h_U\left(u, i_2[v]\right), \end{aligned}$$

as desired. $\square$

Theorem 4.21 *The mapping*

$$\mathcal{J} \,:\, \frac{\mathcal{S}_\Delta(\overline{U})}{\mathcal{H}(X)} \to \left(\frac{\mathcal{S}_\Delta(U)}{\mathcal{H}(X)}\right)'$$

induced by (4.15) is a topological isomorphism of these spaces.

Proof. This follows from Lemma 4.19 and Theorem 4.18.

□

As we have a commutative diagram

$$
\begin{array}{ccccc}
\dfrac{\mathcal{S}_\Delta(\overline{U})}{\mathcal{H}(X)} & \stackrel{i_2}{\hookrightarrow} & \mathcal{S}_\Delta^{(m)}(U) \ominus \mathcal{H}(X) & \stackrel{i_1}{\hookrightarrow} & \dfrac{\mathcal{S}_\Delta(U)}{\mathcal{H}(X)} \\
\cup & & \cup & & \cup \\
\dfrac{\mathcal{S}_A(\overline{U})}{\mathcal{H}(X)} & \stackrel{i_2}{\hookrightarrow} & \mathcal{S}_A^{(m)}(U) \ominus \mathcal{H}(X) & \stackrel{i_1}{\hookrightarrow} & \dfrac{\mathcal{S}_A(U)}{\mathcal{H}(X)}
\end{array}
$$

the pairing $\tilde{h}_U(\cdot,\cdot)$ induces a continuous mapping

$$\mathfrak{J} \,:\, \frac{\mathcal{S}_A(\overline{U})}{\mathcal{H}(X)} \to \left(\frac{\mathcal{S}_A(U)}{\mathcal{H}(X)}\right)' \tag{4.16}$$

which is the restriction of $\mathcal{J}$.

Denote by π the orthogonal projection

$$\pi \,:\, \mathcal{S}_\Delta^{(m)}(U) \ominus \mathcal{H}(X) \to \mathcal{S}_A^{(m)}(U) \ominus \mathcal{H}(X).$$

Corollary 4.22 *The mapping $\mathfrak{J}$ is a topological isomorphism of the spaces (4.16) if and only if*

1) $\mathcal{S}_A(\overline{U})$ *is dense in* $\mathcal{S}_A(U)$;
2) π *maps* $\mathcal{S}_\Delta(\overline{U})$ *continuously into* $\mathcal{S}_A(\overline{U})$.

Proof. According to the general scheme we have

$$
\begin{aligned}
\mathfrak{S}_1 &= \frac{\mathcal{S}_A(U)}{\mathcal{H}(X)}, \\
\mathfrak{S}_2 &= \frac{\mathcal{S}_A(\overline{U})}{\mathcal{H}(X)}, \\
\mathfrak{V} &= \mathcal{S}_A^{(m)}(U) \ominus \mathcal{H}(X),
\end{aligned}
$$

hence the statement follows from Corollary 2.7.

□

Example 4.23 Assume that A is a homogeneous determined elliptic operator with constant coefficients of order m in $X = \mathbb{R}^n$, with $n > 2m$, cf. Example 3.12. As we have already seen in the latter example, the projection π is given by

$$(Mu)(x) = - \int_{\partial U} G_A \left(K_\Phi(x - \cdot), u \right)$$

for any $u \in H^m(U, E)$, where Φ is a fundamental solution of convolution type for A.

If $u \in \mathcal{S}_\Delta(U_\varepsilon) \cap C^m(\overline{U}_\varepsilon, E)$ with $\varepsilon > 0$ small enough, then by the Cauchy-Kovalevskaya Theorem there exist a neighbourhood O of ∂U and a section $v \in C^m(\overline{O}, E)$, such that

$$\begin{cases} Av & = & 0 & \text{in} & O, \\ t(v) & = & t(u) & \text{on} & \partial U. \end{cases}$$

By the definition of M, we get $Mu = Mv$ in U. Then Stokes' formula yields

$$Mu(x) = v(x) - \int_{(\partial O) \cap U} G_A \left(K_\Phi(x - \cdot), v \right)$$

for all $x \in O \cap U$. The right-hand side of the latter equality uniquely extends to a solution of $Au = 0$ in O. Hence we deduce that $Mu \in \mathcal{S}_A(U \cup O)$.

Arguing as in the proof of Lemma 4.2 we see that O is actually independent of u, but it depends on ε and ∂U. Moreover, for every $u \in \mathcal{S}_\Delta(U_\varepsilon) \cap C^m(\overline{U}_\varepsilon, E)$ we have

$$\|v\|_{C^m(\overline{O}, E)} \leq C \, \|u\|_{C^m(\overline{U}_\varepsilon, E)}$$

with $C > 0$ a constant independent of u. Hence M maps $\mathcal{S}_\Delta(\overline{U})$ continuously into $\mathcal{S}_A(\overline{U})$.

Example 4.24 Let A be a determined elliptic operator, as in Example 3.22. Then the condition 1) of Corollary 4.22 is fulfilled. Assume for simplicity that $\mathcal{H}(X) = 0$.

As we have seen in Example 3.22, $\pi = I - T_U \left(A T_U \right)^{-1} A$ where the operator $\left(A T_U \right)^{-1}$ is given by (3.14).

It is well known that the kernel of Φ_{A^*} is real analytic outside of the diagonal $\{x = y\}$. Recall that Φ_{A^*} stands for a two-sided fundamental solution to the operator A^* near $\overline{U}$. Hence it follows that the boundary integral in (3.14) is real analytic in a neighbourhood O of $\overline{U}$.

We thus conclude that $(A T_U)^{-1} f$ is real analytic in O if f has the same property.

If $u \in \mathcal{S}_\Delta(\overline{U}) \cap C^{2m}(\overline{U}_\varepsilon, E)$ with sufficiently small $\varepsilon > 0$, such that $U_\varepsilon \subset\subset O$, then $(A T_U)^{-1} A u$ is real analytic in a neighbourhood of U_ε.

Let us see that $T_U f$ is real analytic in a neighbourhood of $\overline{U}$ if f is real analytic in U_ε.

By the Cauchy-Kovalevskaya Theorem there exist a neighbourhood O' of ∂U and a section $v \in C^m(\overline{O}', E)$, such that

$$\begin{cases} Av = f & \text{in} \quad O', \\ t(v) = 0 & \text{on} \quad \partial U. \end{cases}$$

Then Stokes' formula yields

$$T_U f(x) = T_{U \setminus O'} f(x) + v(x) - \int_{(\partial O') \cap U} G_A(K_{GA^*}(x, \cdot), v)$$

for all $x \in O' \cap U$.

By the definition of v, the right-hand side of the last equality uniquely extends to O' as a real analytic section. Therefore, the same conclusion is valid for $T_U f$.

Finally, arguing as in the proof of Lemma 4.2 one obtains that the neighbourhood O' does not depend on f but does on ε and ∂U. Moreover, for every $f \in C^m(\overline{U}_\varepsilon, E)$ real analytic in U_ε, we get

$$\|v\|_{C^m(\overline{O}', E)} \leq C \, \|f\|_{C^m(\overline{U}_\varepsilon, F)}$$

with $C > 0$ a constant independent of f. Hence, the continuity of π follows from the continuity of the operators A, T_U, H, Φ_{A^*} and formula (3.14).

Example 4.25 Let $X = \mathbb{R}^n$, $n \geq 3$, U be a ball in $\mathbb{R}^n$, and A a Dirac operator in $\mathbb{R}^n$ (cf. Example 3.5). In [Shl96] it is proved that there are systems $\{b_\nu\}$ and $\{c_\mu\}$ of homogeneous harmonic polynomials with values in $\mathbb{C}^k$, such that

a) $\{b_\nu\}$ is an orthogonal basis in all spaces $\mathcal{S}_A^{(1)}(U_\varepsilon)$, where $\varepsilon \in \mathbb{R}$, simultaneously;

b) $\{b_\nu\} \cup \{c_\mu\}$ is an orthogonal basis in all spaces $\mathcal{S}_\Delta^{(1)}(U_\varepsilon)$, $\varepsilon \in \mathbb{R}$, simultaneously.

Property a) implies that condition 1) of Corollary 4.22 is satisfied. Moreover, the projection π restricted to $\mathcal{S}_\Delta^{(1)}(U_\varepsilon)$, $\varepsilon > 0$, coincides with the orthogonal projection

$$\mathcal{S}_\Delta^{(1)}(U_\varepsilon) \to \mathcal{S}_A^{(1)}(U_\varepsilon).$$

It follows that π maps $\mathcal{S}_\Delta(\overline{U})$ continuously into $\mathcal{S}_A(\overline{U})$. Hence Corollary 4.22 is valid for A.

References

[Aiz66]   L. A. AIZENBERG, *The general form of a continuous linear func-tional in spaces of functions holomorphic in convex domains in* $\mathbb{C}^n$, Dokl. Akad. Nauk SSSR **166** (1966), no. 5, 1015–1018.

[AG64]   L. A. AIZENBERG and S. G. GINDIKIN, *On the general form of a linear continuous functional in spaces of holomorphic functions*, Moskov. Oblast. Ped. Inst. Uchen. Zap. **137** (1964), 7–15.

[Aro50]   N. ARONSZAJN, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404.

[Dez80]   A. A. DEZIN, *General Questions of the Theory of Boundary Value Problems*, Nauka, Moscow, 1980.

[Gro53]   A. GROTHENDIECK, *Sur les espaces de solutions d'une classe gen-erale d'equations aux derivees partielles*, J. Anal. Math. **2** (1953), 243–280.

[Hoe83]   L. HÖRMANDER, *The Analysis of Linear Partial Differential Op-erators, V. I*, Springer-Verlag, Berlin et al., 1983.

[Kha66]   V. P. KHAVIN, *Spaces of analytic functions*, In: Mathematical analysis, VINITI, Moscow, 1966, 76–164.

[Koe60]   G. KÖTHE, *Topologische lineare Räume, V. I*, Springer-Verlag, Berlin et al., 1960.

[Mar63]   A. MARTINEAU, *Sur les fonctionelles analytiques et le transforma-tion de Fourier-Borel*, J. Anal. Math. **9** (1963), 1–164.

[Mor93]   M. MORIMOTO, *An Introduction to Sato's Hyperfunctions*, AMS, Providence, Rhode Island, 1993.

[MN57]   C. B. MORREY and L. NIRENBERG, *On the analyticity of the solutions of linear elliptic systems of partial differential equations*, Comm. Pure and Appl. Math. **10** (1957), 271–290.

[NS96]   M. NACINOVICH and A. SHLAPUNOV, *On iterations of the Green integrals and their applications to elliptic differential complexes*, Math. Nachr. **180** (1996), 243–286.

[NST98]   M. NACINOVICH, A. SHLAPUNOV and N. TARKHANOV, *Duality in the spaces of solutions of elliptic systems*, Ann. Scuola Norm. Sup. Pisa **XXVI** (1998), no. 4, 207–232.

[Roi96]   YA. A. ROITBERG, Elliptic Boundary Value Problems in General-
          ized Functions, Kluwer Academic Publishers, Dordrecht NL, 1996.

[Sch60]   M. SCHECHTER, *Negative norms and boundary problems*, Ann.
          Math. **72** (1960), no. 3, 581–593.

[Shl96]   A. SHLAPUNOV, *Spectral decomposition of Green's integrals and
          existence of $W^{s,2}$-solutions of matrix factorizations of the Laplace
          operator in a ball*, Rend. Sem. Mat. Univ. Padova **96** (1996).

[Shl00]   A. SHLAPUNOV *On a dual space for solutions of a Dirac operator
          in ball*, In: Proc. Conf. "Symmetry in Natural Science", Inst. of
          Comp. Mod., Krasnoyarsk, 2000, p. 262–265.

[SST00]   B.-W. SCHULZE, A. SHLAPUNOV, and N. TARKHANOV, *Green
          Integrals on Manifolds with Cracks*, Preprint 2000/12, Univ. Pots-
          dam, Potsdam, 2000, 33 pp.

[ST95]   A. A. SHLAPUNOV and N. TARKHANOV, *Bases with double orthog-
          onality in the Cauchy problem for systems with injective symbols*,
          Proc. London Math. Soc. **71** (1995), no. 3, 1–52.

[Sto95]   E. L. STOUT, *Harmonic duality, hyperfunctions and removable
          singularities*, Izv. RAN, Ser. math. **59** (1995), no. 6, 41 p.

[Str84]   E. J. STRAUBE, *Harmonic and analytic functions admitting a dis-
          tribution boundary value*, Ann. Scuola Norm. Super. Pisa **11** (1984),
          no. 4, 559–591.

[Tar95a] N. TARKHANOV, *Complexes of Differential Operators*, Kluwer Aca-
          demic Publishers, Dordrecht, NL, 1995.

[Tar95b] N. TARKHANOV, *The Cauchy Problem for Solutions of Elliptic
          Equations*, Akademie-Verlag, Berlin, 1995.

[Tar97]   N. TARKHANOV, *The Analysis of Solutions of Elliptic Equations*,
          Kluwer Academic Publishers, Dordrecht, NL, 1997.

[Zna79]   S. V. ZNAMENSKII, *A geometric criterion for strong linear convex-
          ity*, Funkts. Analiz **13** (1979), 224–225.

[Zor82]   P. ZORN, *Analytic Functionals and Bergman spaces*, Ann. Scuola
          Norm. Sup. Pisa **IX** (1982), 365–404.

[Yos65]   K. YOSIDA, *Functional Analysis*, Springer-Verlag, Berlin, 1965.

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