INVESTMENT UNDER UNCERTAINTY WHEN SHOCKS ARE NON-GAUSSIAN

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1. Introduction

The investment literature of the last two decades has recognized the importance of interactions among the irreversibility of investment, uncertainty in the economic environment, and the choice of timing and/or scale of the new investment (see Merton (1971); Pindyck (1982, 1988); Abel (1983); Bertola (1990); Dixit (1993); Dixit and Pindyck (1994); Abel and Eberly (1994); Bertola and Caballero (1994); Metcalf and Hassett (1995); Abel et al. (1996); Caballero and Pindyck (1996); see also the bibliography in Dixit and Pindyck (1994)).

In all of these models, assumptions about the nature of the stochastic processes describing the economic environment are crucial. In the majority of papers, the continuous time stochastic processes are used to model returns or prices. Usually, the (Geometric) Brownian Motion models the movement of variables like the general price level, prices of financial instruments (Fisher (1975)) and option prices (Black and Scholes (1973); Merton (1973)); for a discrete time analog, see Chow (1994). In some cases, the (Geometric) Mean Reverting process is used - see e.g. Dixit and Pindyck (1994); Metcalf and Hassett (1995).

The assumption that the exogeneous variable(s) of interest follow a Brownian motion is very convenient since it allows one to obtain closed form solutions. At the same time, there is some empirical evidence against the modelling of observables as normal random variables. For instance, in many cases distributions with fat tails, in particular, truncated Lévy distributions are observed (see, e.g. Mantegna and Stanley (1995), Cont et al. (1997); these distributions were constructed first by Mantegna and Stanley (1994), and Koponen (1995) suggested a family which admits an explicit description in terms of the Fourier transform). Thus, we

have nice tractable models based on the assumption that observables are normal random variables, and a number of situations when observed processes exhibit fat tails.

In the paper we develop a discrete time model which is almost as tractable as the most popular continuous time models based on the Brownian Motion, yet allows one to treat truncated Lévy flights and more general distributions. We do not make very specific assumptions on the probability distribution. In particular, we do not assume that it is possible to pass to the limit $\Delta t \to 0$ and describe the process by a continuous time model. We believe that this model is especially useful in cases when the time interval between observations is not very small, as it is the case in the theory of real options. (Note that numerical results show that the threshold is sensitive – though not much – to a choice of the time interval).

We apply our model to the Planner's problem (see e.g. Dixit and Pindyck (1994), Ch. 11), and derive an explicit formula for the investment threshold, in terms of an observed distribution density. We produce numerical results for symmetric distributions from a three-parameter Koponen's (1995) family which includes gaussian ones and truncated Lévy distributions.[2] The results show that if a distribution is close to a Lévy distribution, i.e. the truncation happens far from the origin, the threshold increases – in many cases by dozens or even hundreds percent – as we replace a gaussian distribution with the truncated Lévy distribution, of the same variance. If the truncation happens relatively close to the origin, then the threshold may decrease – not significantly, though.

This means that the investment threshold can be made lower by dumping too large fluctuations of economic indices.

The method of the paper is based on a reduction to the Wiener-Hopf equation for the Bellman function[3]. We solve it by the factorization method (Wiener and Hopf (1931)) in a bit more modern form Eskin (1973), assuming that the investment threshold is given. The value function must satisfy certain conditions which lead to an equation for the threshold.

The same approach can be applied to the pricing of the perpetual American put option, under the same very weak assumptions on a probability distribution (see Levendorskiĭ and Boyarchenko (1998)). There also exists a continuous-time version of the method; the corresponding results will be published elsewhere.

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[2]Cont et al. (1997) found that a distribution of this family could be used to describe the Standard & Poor's 500 index futures.

[3]Thus, we see a familiar name though not quite a usual method for the theory of investment

2. The Planner's Problem

The model is a discrete version of the capacity choice model of investment described in Dixit and Pindyck (1994, Ch.11). Consider a planner who chooses investment. The investment is irreversible, and each unit of capital costs K to install. The one period return when Q units of capital are in place is $XU(Q)$ where X is the stochastic shift variable. A discount rate $r > 0$ is fixed. The planner's objective is to maximize the expected present value of returns net of capital installation cost.

Let $x_t = \ln X_t$, and assume that

$$(1) \qquad\qquad x_{t+1} = x_t + \alpha + y_t,$$

where y_t are independently identically distributed random variables with zero mean and the probability distribution density p satisfying

$$(2) \qquad\qquad \int_{-\infty}^{+\infty} p(x) e^x dx < +\infty.$$

For simplicity, we consider symmetric p, though our results admit generalization to the case of non-symmetric distributions. We also need a condition

$$(3) \qquad\qquad q := e^{-r+\alpha} \int_{-\infty}^{+\infty} p(x) e^x dx < 1.$$

Really, for Q constant, the discounted expected returns grow each period by a factor $e^\alpha \int_{-\infty}^{+\infty} p(x) e^x dx$, and are discounted back at rate e^{-r}, and hence are given by

$$w(x) = U(Q) e^x \sum_{j=0}^{+\infty} q^j;$$

for this series to converge, we need (3).

The last condition for p is: there exist C, ϵ_0 and $\omega > 0$ such that $\hat{p} = \mathcal{F}p$, the Fourier transform of p, and its derivative satify bounds

$$(4) \qquad\qquad \hat{p}(k) \leq 1, \ \forall \ k \in R,$$

$$|\hat{p}(k)| + |\hat{p}'(k)| \leq C(1 + |k|)^{-\omega}, \ \forall \ k, \ |\Im k| \leq \epsilon_0.$$

(Due to (2), $\hat{p}$ is holomorphic on a strip $|\Im k| < 1$, therefore (4) makes sense). The second bound is a weak form of a smoothness condition. For instance, for a piece-wise smooth p it holds with $\omega = 1$.

Let $w(Q, x)$ be the Bellman function. Due to the absence of variable cost,

$$(5) \qquad\qquad w(Q, x) \text{ is non-decreasing w.r.t. } Q, \text{ for } x \text{ fixed},$$

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and clearly,

(6) $\qquad$ $w(Q, x)$ is non-decreasing w.r.t. x, for Q fixed,

and

(7) $\qquad$ $w(Q, x)$ is non-negative.

Due to (6)–(7),

(8) $\qquad$ $w(Q, x)$ is measurable and locally integrable w.r.t. x.

(If $w(Q, x) = +\infty$ for $x \geq b$, then for $a < b < c$, the integral over (a, c) is $+\infty$). Finally, assume that

(9) $\qquad$ U is differentiable and concave.

An argument on p.p. 360–361 in Dixit and Pindyck (1994) shows that (9) imply

(10) $\qquad$ $w(Q, x)$ is concave w.r.t. Q, $\quad$ for x fixed.

In discrete time, the Bellman equation for the problem under consideration is

$$(11)\, w(Q, x_{j-1}) = \max_{Q' \geq Q} \{\exp(x_{j-1}) U(Q) - K(Q' - Q) + e^{-r} E[w(Q', x_j)|x_{j-1}]\},$$

where E is the expectation operator. Suppose that a point (Q, x) is in the inaction region, i.e. the maximum in (11) is attained at $Q' = Q$. Then

$$w(Q, x) = U(Q)e^x + e^{-r} E[w(Q', x_j)|x_{j-1}],$$

or, on the strength of (1),

$$(12) \qquad w(Q, x) = U(Q)e^x + e^{-r} \int_{-\infty}^{+\infty} p(y) w(Q, x + \alpha + y) dy =$$

$$= U(Q)e^x + e^{-r} \int_{-\infty}^{+\infty} p(x + \alpha - y) w(Q, y) dy,$$

for all $x < h(Q)$, where $x = h(Q)$ is the boundary of the inaction region.

$\quad$ **Lemma 2.1.** *Let (1), (2) and (9) hold, and let there exist (Q, x) such that* $w(Q, x) < +\infty$.

$\quad$ *Then $w(Q, x) < +\infty$ for all (Q, x).*

$\quad$ **Proof.** Suppose that for some y, $w(Q, y) = +\infty$. Then, on the strength of (6), $w(Q, z) = +\infty$, $\forall z > y$, and the RHS in (12) is infinite. The contradiction shows that $w(Q, x) < +\infty$ $\forall x$.

Due to (5), for $Q_1 < Q$, $w(Q_1, x) < +\infty$, and we may assume that Q_1 is in the action region. If (Q, x) and (Q_1, x) are in the action region,

(13) $\quad w(Q, x) - w(Q_1, x) = K(Q - Q_1) + (U(Q) - U(Q_1))e^x, \quad \forall\, x > h(Q).$

By dividing (13) by $Q - Q_1$ and passing to the limit $Q_1 \to Q$, we see that in the action region $w_Q(Q, x)$ exists and

(14) $\qquad\qquad\qquad w_Q(Q, x) = K + U'(Q)e^x, \quad x > h(Q).$

It follows from (9), (10) and (14), that there exist $C = C(Q_1, x)$ such that for all $Q > Q_1$, $w_Q(Q, x) < C$. By integrating, we obtain $w(Q, x) < +\infty$.

Lemma has been proved.

The disinvestment is never optimal since there is no variable cost and the installation cost cannot be recovered should X fall very low. Hence, h is non-decreasing, and for almost all Q, the derivative

(15) $\qquad\qquad\qquad\qquad h'(Q) \text{ exists.}$

Below, we consider only Q satisfying (15) and derive a formula for $h(Q)$. We will see that the expression obtained defines a continuous function, therefore the formula will be valid for all Q.

Due to (4), for a given x, $w_Q(Q, x)$ exists for almost all Q, and by (14), for (Q, x) in the action region; in Appendix, we show that if Q satisfies (15), $w_Q(Q, x)$ exists for almost all $x < h(Q)$. We choose Q satisfying (15), and differentiate (12) w.r.t. Q:

(16) $\quad w_Q(Q, x) = U'(Q)e^x + e^{-r} \displaystyle\int_{-\infty}^{+\infty} p(x + \alpha - y)w_Q(Q, y)dy, \;\; \forall\, x < h(Q).$

Due to (2), $\hat{p}(k)$ is holomorphic on a strip $|\Im k| < 1$ and is continuous on the closed strip $|\Im k| \leq 1$. Set

$$A(k) = 1 - e^{-r + i\alpha k}\hat{p}(k),$$

and define an operator $A(D)$ by

$$(A(D)u)(x) = (\mathcal{F}^{-1}A(k)\mathcal{F}u)(x) = (2\pi)^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(x-y)}A(k)u(y)dydk.$$

$A(D)$ is called a pseudo-differential operator (PDO) with the symbol $A(k)$. Note that $D = -i\partial/\partial x$, and the Taylor formula gives $u(x + \alpha) = (\exp(i\alpha D)u)(x)$. Using this equality and an equality

$$\int_{-\infty}^{+\infty} p(x - y)u(y)dy = (\mathcal{F}^{-1}\hat{p}(k)\mathcal{F}u)(x),$$

we may rewrite (16) as

$$(17) \qquad\qquad (A(D)w_Q)(Q,x) = U'(Q)e^x, \quad x < h(Q).$$

Fix Q and h, a prospective kandidat for $h(Q)$, and set

$$U' = U'(Q), \quad u(x) = w_Q(Q, x+h) - K - U'e^{x+h}.$$

Due to (2), for $|\beta| \leq 1$,

$$A(D)e^{\beta x} = (2\pi)^{-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(x-y)k} A(k) e^{\beta y} dy dk = A(-i\beta)e^{\beta x},$$

therefore in terms of u, (14) and (17) are

$$(18) \qquad\qquad u(x) = 0, \ x > 0,$$

and
$$(19) \qquad\qquad (A(D)u)(x) = qU'e^{x+h} - r_1 K, \quad x < 0,$$

where $q = 1 - A(-i)$ is the same as in (3), and $r_1 = A(0) = 1 - e^{-r}$.

Take small $\epsilon \in (0,1)$, and set

$$u^\epsilon(x) = e^{\epsilon x} u(x), \quad A^\epsilon(k) = A(k + i\epsilon).$$

By multiplying (18)–(19) by $e^{\epsilon x}$ and taking into account that $e^{\epsilon x} A(D) e^{-\epsilon x} = A^\epsilon(D)$, we obtain

$$(20) \qquad\qquad (A^\epsilon(D)u^\epsilon)(x) = qU'e^h e^{(1+\epsilon)x} - r_1 K e^{\epsilon x}, \quad x < 0,$$

$$(21) \qquad\qquad u^\epsilon(x) = 0, \quad x > 0.$$

To solve (20)–(21), we need the following lemma. It is a variant of standard factorization theorems (see e.g. Eskin (1973), Section 6).

Lemma 2.2. *Let (2) and (4) hold.*

Then there exists $\epsilon_0 > 0$ such that for any $|\epsilon| \leq \epsilon_0$, $A^\epsilon(k)$ admits a factorization
$$(22) \qquad\qquad A^\epsilon(k) = A_+^\epsilon(k) A_-^\epsilon(k)$$

with the $A_\pm^\epsilon(k)$ satisfying the following conditions:

a) A_+^ϵ (resp. A_-^ϵ) is holomorphic in a half-plane $\Im k > 0$ (resp. $\Im k < 0$), and admits a continuous extension into the closed half-plane;

b) there exist $c > 0, C$ such that

$$(23) \qquad\qquad c \leq |A_\pm^\epsilon(k)| \leq C, \quad \forall \ \pm\Im k \geq 0;$$

c) $A_\pm^\epsilon(k)^{-1}$ *admits a representation*

(24) $$A_\pm^\epsilon(k)^{-1} = 1 + T_\pm^\epsilon(k),$$

where $T_\pm^\epsilon$ *is holomorphic in a half-plane* $\pm \Im k > 0$, *and satisfies an estimate*

(25) $$|T_\pm^\epsilon(k)| \le C(1 + |k|)^{-\omega_1}, \quad \forall \pm \Im k \ge 0,$$

where C *and* $\omega_1 > 0$ *are independent of* k.

Proof. For $\epsilon = 0$, $\Re A(l) \ge 1 - e^{-r} > 0$, $\forall l \in R$ (see (4)), and on the strength of (2) and (4), if $|\epsilon|$ is not too large, then there exists $c_\epsilon > 0$ such that $\Re A^\epsilon(l) \ge c_\epsilon$, $\forall l \in R$. For such ϵ and $l \in R$, $\ln A^\epsilon(l)$ is well defined by a requirement: $\ln a$ is real for $a > 0$, and we may set, for $\tau > 0$ and $k \in R$,

(26) $$b_\pm^\epsilon(k \pm i\tau) = \pm \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\ln A^\epsilon(l)}{k \pm i\tau - l} dl,$$

$$A_\pm^\epsilon(k \pm i\tau) = \exp(b_\pm^\epsilon(k \pm i\tau)).$$

The proof that $A_\pm^\epsilon$ satisfy (22) and a)–c) is a minor variation of the proof in Eskin (1973); for completeness, we give it in Appendix.

Parts a) and b) of Lemma 2.2 allow one to obtain a unique solution $u^\epsilon \in L_2(R_-)$ to a problem (20)–(21) (see e.g. Theorem 7.1 in Eskin (1973)):

$$u^\epsilon = A_+^\epsilon(D)^{-1} \theta_- A_-^\epsilon(D)^{-1}(qU'e^h e^{(1+\epsilon)x} - r_1 K e^{\epsilon x}),$$

where θ_- is the characteristic function of R_-. By multiplying by $e^{-\epsilon x}$, we obtain

(27) $$u = e^{-\epsilon x} A_+^\epsilon(D)^{-1} \theta_- A_-^\epsilon(D)^{-1}(qU'e^h e^{(1+\epsilon)x} - r_1 K e^{\epsilon x}).$$

The derivation of a formula for $h = h(Q)$ is based on an analysis of the behaviour of $u(x) = u(x, h)$ near zero.

Lemma 2.3. *For* $x > 0$, $u(x, h) = 0$, *and as* $x \to -0$,

(28) $$u(x, h) = d(h)(1 + e^h \mu_1(x) + \mu_2(x)) +$$

$$+ xA_-^0(0)^{-1} r_1 K + e^h \chi_1(x) + \chi_2(x),$$

where functions $\mu_1(x) = o(1), \mu_2(x) = o(1), \chi_1(x) = o(x), \chi_1(x) = o(x)$ *are independent of* h, *and* $d(h) = A_-^0(-i)^{-1} qU'e^h - A_-^0(0)^{-1} r_1 K$.

Proof. The first statement is just (19), and (28) will be proved in Appendix. By returning to $w_Q(Q, x)$, we obtain

(29) $$w_Q(Q, x, h) = K + U'(Q)e^x + d(h)(1 + e^h \mu_1(x - h) + \mu_2(x - h)) +$$

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$$+(x - h)d(h)A_-^0(-i)^{-1}qU'(Q)e^h + e^h\chi_1(x - h) + \chi_2(x - h),$$

as $x \to h - 0$. Direct calculations (see Appendix) show that $b_-^0(-i)$ and $b_-^0(0)$ are real, therefore $A_-^0(-i)$ and $A_-^0(0)$ are positive, and $d(h)$ is real. Eq. (29) implies

$$\lim_{x \to h-0} w_Q(Q, x, h) = K + U'(Q)e^h + d(h),$$

and since

$$\lim_{x \to h+0} w_Q(Q, x, h) = K + U'(Q)e^h,$$

an assumption $d(h) \neq 0$ contradicts (10).

If $d(h) = 0$, (29) gives

$$\lim_{x \to h-0} w_Q(Q, x, h) = K + U'(Q)e^h,$$

and

$$\lim_{x \to h-0} w_{Qx}(Q, x, h) = A_-^0(0)^{-1}r_1 K + U'(Q)e^h >$$

$$> U'(Q)e^h = \lim_{x \to h+0} w_{Qx}(Q, x, h),$$

which agrees with (10) but shows that the smooth pasting condition (valid for a gaussian continuous time model) fails in our discrete time model.

Clearly, $d(h) = 0$ if and only if

$$(30) \qquad\qquad H(Q) = e^{h(Q)} = \frac{r_1 A_-^0(-i)}{A_-^0(0)qU'}K,$$

and direct calculations (see Appendix) show that

$$(31) \qquad\qquad A_-^0(-i)/A_-^0(0) = r_1^{-1/2}\exp(I_1 - I_2),$$

where

$$I_1 = \frac{1}{2\pi}\int_0^{+\infty} \ln((1 - e^{-r}\hat{p}(l)\cos(\alpha l))^2 + (e^{-r}\hat{p}(l)\sin(\alpha l))^2)(1 + l^2)^{-1}dl,$$

$$I_2 = \frac{1}{\pi}\int_0^{+\infty} \arctan\left(\frac{\hat{p}(l)\sin(\alpha l)}{e^r - \hat{p}(l)\sin(\alpha l)}\right) l^{-1}(1 + l^2)^{-1}dl.$$

Theorem 2.1. *Let (1) – (4) hold.*

Then the investment threshold is given by (30)–(31).

Proof. We have proven (30) for almost all Q. Since h is non-decreasing and the RHS in (30) is continuous, (30) holds for all Q.

To facilitate the comparison with the Marshallian prescription, we rewrite (30) as

$$(32) \qquad \frac{H(Q)U'(Q)}{1-q} = \frac{A^0_-(-i)r_1}{A^0_-(0)q(1-q)}K.$$

$XU'(Q)$ is the marginal utility, and the expected present value of it is equal to $XU'(Q)/(1-q)$ – see a discussion after (3). A textbook Marshallian calculation tells the planner to invest when this value exceeds the cost K, but, as in Dixit and Pindyck (1994), an additional factor

$$\kappa = \frac{A^0_-(-i)r_1}{A^0_-(0)q(1-q)}$$

intervenes; in Dixit in Pindyck (1994), the factor is $\kappa_0 = \beta/(\beta - 1)$, where $\beta > 1$ is a positive root to the characteristic equation $k^2\sigma^2/2 + \alpha k^2 - r = 0$.

Here the factor, κ, is rather complicated, and it is difficult to perform a comparative statics analysis of it; still, it is not difficult to calculate it numerically.

Numerical Examples. The first truncated Lévy distributions were constructed by Mantegna and Stanley (1994). Later, Koponen (1995) constructed a family of truncated Lévy distributions which admit explicit description in terms of their Fourier transforms. For the sake of brevity, we consider only symmetric distributions of this family, with $\hat{p}_\nu$ defined by

$$\hat{p}_\nu(k) = \exp[-\sigma^2\lambda^2[1 - ((k/\lambda)^2 + 1)^{\nu/2}\cos(\nu \arctan(k/\lambda))]/\nu(\nu - 1)],$$

where $\sigma > 0, \lambda > 0$ and $\nu \in (0, 2], \nu \neq 1$ are parameters. We have chosen a normalization so that the variance is independent of ν and λ.

For $\nu = 2$, we obtain $\hat{p}_2(k) = \exp(-\sigma^2 k^2/2)$ which means that p_2 is a gaussian distribution. As ν moves from 2 down, p_ν deviates from a gaussian distribution, and for fixed $\nu \in (0, 2), \nu \neq 1$, in the limit $\lambda \to +0$, p_ν becomes a Lévy distribution with $\hat{p}_\nu(k) = \exp(-c_1|k|^\nu \cos(\nu\pi/2)/\nu(\nu - 1))$. Roughly speaking, $(-\lambda^{-1}, \lambda^{-1})$ is an interval where p_ν differs insignificantly from a Lévy distribution, and for $|x| >> \lambda^{-1}$, the distribution exhibits an exponential fall-off.

Here are some numerical examples.[4] In tables below, we fix $r, \alpha, \sigma, \lambda$ with and see how the factor κ varies with ν. Since Δt is normalized to unity, r, α, σ have to be small which explains choices in examples below. κ_0, the factor in Dixit and Pindyck (1994), is independent of λ and ν.

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[4]The authors thank Mitya Boyarchenko for the help with calculations

Table 1. *Parameters: $r = 0.006$, $\alpha = -0.002$, $\sigma = 0.095$, $\lambda = 1.5$* $\kappa_0 = 3.512$.

ν	2.0	1.8	1.6	1.4	1.2	0.8	0.6	0.4	0.2
κ	3.332	3.341	3.358	3.387	3.431	3.558	3.723	3.922	4.223

Table 2. *Parameters: $r = 0.006$, $\alpha = -0.002$, $\sigma = 0.111$, $\lambda = 1.5$* $\kappa_0 = 7.011$.

ν	2.0	1.8	1.6	1.4	1.2	0.8	0.6	0.4	0.2
κ	6.578	6.732	6.973	7.334	7.873	10.029	12.428	17.731	37.637

In these two examples, it is clearly seen that the factor κ grows as ν goes from 2 down, i.e. as a process deviates from a gaussian one of the same variance.

In Example 1, the factor κ can grow by more than 26%, and in Example 2, by 572%.

The following example shows that the factor can decrease, though not that significantly – by 4.5%; as compared with the continuous time model – by 8.7%

Table 3. *Parameters: $r = 0.006$, $\alpha = -0.002$, $\sigma = 0.079$, $\lambda = 2$* $\kappa_0 = 2.348$.

ν	2.0	1.8	1.6	1.4	1.2	0.8	0.6	0.4	0.2
κ	2.252	3.242	2.231	2.218	2.204	2.173	2.159	2.150	2.145

We see that the factor can increase quite dramatically as we replace a gaussian distribution by a non-gaussian one with the same variance. There are also cases when the factor decreases, though not significantly. This happens if a distribution is obtained from a Lévy one by a truncation too close to origin.

3. Conclusion

In the paper, we have constructed a discrete time model of investment under uncertainty, which is applicable in the case of non-gaussian distributions. This model admits a closed form solution as the standard continuous time model based on the Geometric Brownian Motion (systematically used by Dixit and Pindyck (1994) and other authors) does. Our model allows one to treat more complex processes and does not require that the passing to the continuous time limit be possible. The last remark is essential in applications where the time increment is not very small. In addition, in cases when the underlying stochastic process is a mixture of continuous and jump processes, our model does not require the separation of the mixture, as standard models do.

Here is another characterization of our model: in standard models of irreversible investment under uncertainty, only information about mean and variance is used whereas in our model – about moments of higher order as well.

According to standard models, the volatility changes the threshold for the investment: the higher the volatility of the price of a commodity (and the standard measure of the volatility is the variance), the higher level of the price is needed to trigger the new investment. Our model shows that in the case of fat-tailed distributions, the threshold depends on the higher moments, and in some cases can be much higher still: it can grow with the higher moments even if the variance remains the same. However, there are also cases when the threshold decreases, and this may happen when fat tails are truncated in a small vicinity of the origin. This implies that policy interventions should aim at dumping of large fluctuations rather than at decreasing of the average volatility, i.e. variance.

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Appendix

A1. Some basic facts of the theory of the Sobolev spaces and the theory of pseudo-differential operators (see e.g. Eskin (1973)

By $\mathcal{S}(R)$ one denotes the space of infinitely differentiable functions decaying at infinity faster than any power of x, together with all derivatives, and by $\mathcal{S}'(R)$ – its dual space.

Let $s \in R$. The Sobolev space $H^s(R)$ consists of $u \in \mathcal{S}'(R)$ with the finite norm

$$\| u \|_s = \left(\int_{-\infty}^{+\infty} (1 + k^2)^s |\hat{u}(k)|^2 dk \right)^{1/2}.$$

The closure of $C_0^\infty(R_\pm)$ in $H^s(R)$ is denoted by $\overset{o}{H}{}^s(R_\pm)$. The spaces $H^s(R)$ and $\overset{o}{H}{}^s(R_\pm)$ are Hilbert spaces, and $H^0(R) = L_2(R)$, $\overset{o}{H}{}^0(R_\pm) = L_2(R_\pm)$.

For an integer $m \geq 0$ and $s > m + 1/2$, $H^s(R) \subset C^m(R)$, by the Sobolev embedding theorem.

The Dirac delta-function (a linear functional defined by $\delta(f) = f(0)$) belongs to $\overset{o}{H}{}^s(R_\pm)$, for any $s < -1/2$.

If the symbol of a PDO $A(D)$ is measurable and admits a bound

$$(33) \qquad\qquad |A(k)| \leq C(1 + |k|^2)^{m/2}, \ \forall \ k,$$

then $A(D)$ is said to be of order m. A PDO of order m is a bounded operator from $H^s(R)$ to $H^{s-m}(R)$. If $A(k)$ admits a holomorphic extension into a half-plane $\pm \Im k > 0$ and satisfies a bound (33) in the closed half-plane, then $A(D)$ is a bounded operator from $\overset{o}{H}{}^s(R_\mp)$ to $\overset{o}{H}{}^l(R_\mp)$, where $l = s - m$.

Let J be an interval, and T be a PDO of negative order. Then for $f \in C^\infty(J) \cap \mathcal{S}'(R)$, a solution $u \in \mathcal{S}'(R)$ to an equation $(I - T)u = f$ also is C^∞ on J.

For T of arbitrary order and $f \in C^\infty(J) \cap \mathcal{S}'(R)$, Tf is C^∞ on J.

A2. The existence of w_Q. Let Q satisfy (13). To prove that $w_Q(Q, x)$ exists for almost all $x < h(Q)$, note that (12) can be written in the form

$$((I - T)w)(Q, x) = U(Q)e^x, \ \forall \ x < h(Q),$$

where by (4), T is a PDO of negative order $-\omega$. The RHS being of the class C^∞, a solution $w \in C^\infty((-\infty, h(Q)))$. Since w is non-decreasing and bounded on $(-\infty, h(Q))$, it admits a continuous extention on $(-\infty, h(Q)]$.

Now fix $Q_1 > Q$ and set $\tilde{w}(Q, Q_1, x) = w(Q, x) - w(Q_1, x)$. By substituting into (12) and (14), we obtain

$$((I - T)\tilde{w})(Q, Q_1, x) =$$

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$$= q(U(Q) - U(Q_1))e^x - A(0)K \cdot (Q - Q_1) - w(Q_1, x) + (Tw)(Q_1, x),$$

for $x < h(Q)$, and

$$\tilde{w}(Q, Q_1, x) = 0, \ x > h(Q),$$

where T is an integral operator with the positive kernel $k(x-y) = e^{-r}p(x+\alpha-y)$. Let e_h, $h = h(Q)$, be the extension by zero operator from $(-\infty, h)$ on R, and r_h the corresponding restriction operator, and set $T_h = r_h T e_h$. Then the problem above can be written in the form of the Wiener-Hopf equation in $L_\infty((-\infty, h))$:

$$(I - T_h)\tilde{w} = \sum_j f_j(Q)u_j,$$

where $f_j \in C^1$, and $u_1(x) = e^x, u_2(x) = 1, u_3(x) = -w(Q_1, x), u_4(x) = Tw(Q_1, x)$ are independent of Q and belong to $C((-\infty, h])$.

The operator norm of T_h as an operator in $L_\infty((-\infty, h))$ is

$$\parallel T_h \parallel = \parallel k \parallel_{L_1(R)} = e^{-r} < 1,$$

and hence, $I - T_h$ is invertible with the inverse $(I - T_h)^{-1} = I + T_h + T_h^2 + T_h^3 + \cdots$. By invoking the theorem on the differentiability of the integral w.r.t. the upper limit, we see that functions $v_j(x, h) = ((I - T_h)^{-1}u_j)(x)$ are differentiable w.r.t. h. It follows that if (13) holds, then $\tilde{w}(Q, Q_1, x) = \sum_j f_j(Q)v_j(x, h(Q))$ is differentiable w.r.t. Q, for $x < h(Q)$. Since $w(Q_1, x)$ is independent of Q, $w(Q, x)$ is also differentiable w.r.t. Q, for $x < h(Q)$.

A3. Proof of Lemma 2.2. Fix ϵ, for which functions in (26) are well-defined, and suppress an index ϵ. As $l \to \pm\infty$, $\ln A(l) = O(\hat{p}(l))$, and on the strength of (4), there exist $C, \omega > 0$ such that

$$(34) \qquad\qquad |\ln A(l)| \le C(1 + |l|)^{-\omega}, \ \forall \ l \in R.$$

Fix $\tau_0 > 0$, and consider $b_\pm(k)$ in a half-plane $\pm\Im k \ge \tau_0$. Set $J_1 = \{l \mid |k - l| \ge |k|/2\}$, $J_2 = \{l \mid |k-l| \le |k|/2\}$. On J_1, $|k| \le 2|l-k|$ and $|l| \le |k-l| + |k| \le 3|l-k|$, hence

$$(35) \quad (1 + |k - l|)^{-1}(1 + |l|)^{-\omega} \le 3(1 + |k|)^{-\omega/2}(1 + |l|)^{-1+\omega/2}(1 + |l|)^{-\omega},$$

and since $(1 + |l|)^{-1-\omega/2} \in L_1(R)$, we deduce from (34)–(35) an estimate

$$(36) \qquad\qquad \left| \int_{J_1} \frac{\ln A(l)}{k - l} dl \right| \le C_{\tau_0}(1 + |k|)^{-\omega/2},$$

where a constant C_{τ_0} is independent of k and $\tau \ge \tau_0$. On J_2, $|l| \ge |k| - |k - l| \ge |k|/2 \ge |k - l|$, and hence,

$$(1 + |k - l|)^{-1}(1 + |l|)^{-\omega} \le C(1 + |k - l|)^{-1}(1 + |k - l|)^{-\omega/2}(1 + |k|)^{-\omega/2}.$$

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Therefore,

$$\left| \int_{J_2} \frac{\ln A(l)}{k-l} dl \right| \leq$$

$$\leq C_{1\tau_0} (1 + |k|)^{-\omega/2} \int_{-\infty}^{+\infty} (1 + |k-l|)^{-1-\omega/2} dl \leq C_{2\tau_0} (1 + |k|)^{-\omega/2}.$$

Thus, (36) holds with R instead of J_1. Similar estimates hold for derivatives w.r.t. k, and in a region $\pm\Im k \geq \tau_0$, parts b), c) and the first part of a) have been proved.

To show that $A_\pm(k)$ admits a continuous extension up to the boundary of a half-plane $\pm\Im k > 0$, fix $k \in R$ and write, for $\tau > 0$,

$$b_\pm(k \pm i\tau) = b_\pm^1(k \pm i\tau) + b_\pm^2(k \pm i\tau),$$

where

$$b_\pm^1(k \pm i\tau) = \pm\frac{i}{2\pi} \int_{|k-l|>1} \frac{\ln A(l)}{k \pm i\tau - l} dl,$$

$$b_\pm^2(k \pm i\tau) = \pm\frac{i}{2\pi} \int_{|k-l|<1} \frac{\ln A(l)}{k \pm i\tau - l} dl.$$

The denominator of the integrand of $b_\pm^1$ being bounded away from zero, uniformly in $k \in R$ and $\tau \geq 0$, the proof for $\pm\Im k \geq \tau_0$ above shows that $b_\pm^1(k)$ is continuous in a closed half-plane $\pm\Im k \geq 0$ and satisfies all the necessary estimates there.

Consider $b_\pm^2(k)$. On the strength of (4), there exists $C > 0$ such that for all $k \in R$ and $c \in (k-1, k+1)$,

$$|\ln A(c)| + |A'(c)/A(c)| \leq C(1 + |k|)^{-\omega}.$$

Hence, using the Lagrange formula

$$\ln A(l) = \ln A(k) + \frac{A'(c)}{A(c)}(l - k),$$

where $c \in (k, l)$ (or $c \in (l, k)$), and noticing that

$$\pm\frac{i}{2\pi} \int_{|k-l|<1} \frac{dl}{k \pm i\tau - l} = \pm\frac{i}{2\pi} \int_{|l|<1} \frac{dl}{i\tau - l} =$$

$$= \pm\frac{i}{2\pi} \int_{|l|<1} \frac{\mp i\tau - l}{l^2 + \tau^2} dl = \frac{1}{2\pi} \int_{|l|<1} \frac{\tau dl}{l^2 + \tau^2} \to 1/2,$$

as $\tau \to +0$, we obtain that $b_\pm^2(k)$ is continuous up to the boundary of a half-plane $\pm\Im k \geq 0$, and admits an estimate

$$|b_\pm^2(k \pm i\tau)| \leq C(1 + |\tau| + |k|)^{-\omega/2}.$$

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This finishes the proof of Lemma 2.2.

A3. Proof of Eq. (28). By the residue theorem, we have, for $\beta > 0$ and $\tau > \beta$,

$$A_-^\epsilon(D)\theta_- e^{\beta x} = (2\pi)^{-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{0} e^{i(x-y)k} A_-^\epsilon(k) e^{\beta y} dy dk =$$

$$= (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ixk} A_-^\epsilon(k)(-ik+\beta)^{-1} dk =$$

$$= (-2\pi i)^{-1} \int_{-\infty-i\tau}^{+\infty-i\tau} e^{ixk} A_-^\epsilon(k)(k+i\beta)^{-1} dk + (-2\pi i)^{-1}(-2\pi i) A_-^\epsilon(-i\beta).$$

The first term in the RHS tends to 0 as $\tau \to +\infty$, therefore

(37) $$A_-^\epsilon(D)\theta_- e^{\beta x} = A_-^\epsilon(-i\beta) e^{\beta x}, \ \forall \ x < 0,$$

and similarly, $A_-^\epsilon(D)\theta_- e^{\beta x} = 0, \ \forall \ x > 0$. Using (26), Lemma 2.2 a) and the residue theorem, we derive obtain:

(38) $$A_-^\epsilon(-i(1+\epsilon)) = -\frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{A^\epsilon(l)}{k - i(1+\epsilon) - l} dl =$$

$$= -\frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{A(l+i\epsilon)}{k - i(1+\epsilon) - l} dl = -\frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{A(l)}{k - i - l} dl = A_-^0(-i),$$

and due to (4),

(39) $$A_-^\epsilon(-i\epsilon) = -\frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{A^\epsilon(l)}{k - i\epsilon - l} dl =$$

$$= -\frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{A(l+i\epsilon)}{k - i\epsilon - l} dl = A_-^0(0).$$

Substitute (37)– (39) into (27); the result is

(40) $$u = e^{-\epsilon x} A_+^\epsilon(D)^{-1} e^{\epsilon x} \theta_- f,$$

where $f(x) = A_-^0(-i)^{-1} q U' e^h e^x - A_-^0(0) r_1 K$. By (2), $A(k)$ is holomorphic on a strip $|\Im k| < 1$, and bounded away from zero on the real axis, hence, by (4), $A(k+i\epsilon)$ also is bounded away from zero on the real axis, provided ϵ is sufficiently small. Using these observation and (37), we obtain, similarly to (38) – (39),

$$e^{-\epsilon x} A_+^\epsilon(D)^{-1} e^{\epsilon x} = e^{-\epsilon x} A^\epsilon(D)^{-1} e^{\epsilon x} e^{-\epsilon x} A_-^\epsilon(D) e^{\epsilon x} =$$

$$= A^0(D)^{-1} A_-^0(D) = A_+^0(D).$$

Hence, we may rewrite (40) as

(41) $$u = A_+^0(D)^{-1} \theta_- f.$$

By using (26), it is straightforward to show that $A_+^0(k)$ is real for real k, and $A_-^0(-i)$, $A_-^0(0)$ are real as well. Hence, u is real-valued (and independent of small $\epsilon > 0$). Nevertheless, to ensure the applicability of various results of the theory of pseudo-differential operators, it is more convenient to proceed with (40) in the form

(42)
$$u = e^{-\epsilon x} A_+^\epsilon(D)^{-1} \theta_- f^\epsilon,$$

where $f^\epsilon(x) = A_-^0(-i)^{-1} q U' e^h e^{(1+\epsilon)x} - A_-^0(0) r_1 K e^{\epsilon x}$. Since $f^\epsilon \in \mathcal{S}(\overline{R_-})$, we can apply a formula (5.38) in Eskin (1973):

(43) $\theta_- f^\epsilon = \sum_{s=1}^m (1 - iD)^{-s} \delta \cdot ((1 - iD)^{s-1} f^\epsilon)(0) + (1 - iD)^{-m} \theta_- (1 - iD)^m f^\epsilon.$

Here m is a positive integer, δ is the Dirac delta-function, and

$$((1 - iD)^{-s} \delta)(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ixk}(1 - ik)^{-s} dk.$$

By using (43) and (24)–(25), we can rewrite (42) as

$$u = e^{-\epsilon x}(1 + T_+^\epsilon(D))[(1 - iD)^{-1} \delta \cdot f^\epsilon(0) +$$

$$+ (1 - iD)^{-2} \delta \cdot ((1 - iD) f^\epsilon)(0) + (1 - iD)^{-2} \theta_- (1 - iD)^2 f^\epsilon].$$

By introducing the notation

$$\phi_s = (1 - iD)^{-s} \delta, \quad w_1^\epsilon = T_+^\epsilon(D)(1 - iD)^{-1} \delta,$$

$$w_2^\epsilon = A_+^\epsilon(D)^{-1}(1 - iD)^{-2} \delta,$$

$$w_3^\epsilon = A_+^\epsilon(D)^{-1}(1 - iD)^{-2} A_-^0(-i)^{-1} q U' \theta_- ((1 - iD)^2 e^{(1+\epsilon)x}),$$

$$w_4^\epsilon = -A_+^\epsilon(D)^{-1}(1 - iD)^{-2} A_-^0(0)^{-1} r_1 K \theta_- ((1 - iD)^2 e^{\epsilon x}),$$

we can write

(44)
$$u = e^{-\epsilon x} \{ (\phi_1 + w_1^\epsilon) f^\epsilon(0) + \phi_2 \cdot (f^\epsilon(0) - (f^\epsilon)'(0)) +$$

$$+ w_2^\epsilon \cdot (f^\epsilon(0) - (f^\epsilon)'(0)) + e^h w_3^\epsilon + w_4^\epsilon \}.$$

Consider terms in (44).

1) We know that $\delta \in \overset{o}{H}{}^l (R)$, for any $l < -1/2$, and since $f^\epsilon \in \mathcal{S}(\overline{R_-})$, Theorem 5.1 in Eskin (1973) gives $\theta_- f^\epsilon, \theta_- (1 - iD)^2 f^\epsilon \in H^s(R)$, for any $s \in (0, 1/2)$. But $T_+^\epsilon(D)(1 - iD)^{-1}$ and $A_+^0(D)(1 - iD)^{-2}$ are PDO of order $-1 - \omega$ and -2, respectively, and hence, $w_1^\epsilon \in H^{l+1+\omega}(R) \subset H^{1/2+\omega/2}(R) \subset C(R)$, $w_2^\epsilon \in H^{l+2}(R) \subset C^1(R)$. Similarly, $w_3^\epsilon, w_4^\epsilon \in C^1(R)$. The symbols of $A_+^\epsilon(D)^{-1}$ and

$T^\epsilon_+(D)$ being holomorphic in a half-plane $\Im k > 0$, we have $\operatorname{supp} w^\epsilon_j \subset (-\infty, 0]$, $j = 1, \cdots, 4$. Hence, $w^\epsilon_1(0) = 0$, and $w^\epsilon_j(0) = (w^\epsilon_j)'(0) = 0$, $j = 2, 3, 4$.

2) Consider ϕ_s, $s = 1, 2$. By the residue theorem, for $x > 0$ and $\tau > 0$,

$$\phi_s(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ixk}(1 - ik)^{-s} dk = (2\pi)^{-1} \int_{-\infty+i\tau}^{+\infty+i\tau} e^{ixk}(1 - ik)^{-s} dk \to 0,$$

as $\tau \to +\infty$, and hence, $\phi_s(x) = 0$. Further,

$$\int_{-\infty}^0 e^{-ixk} e^x dx = (1 - ik)^{-1},$$

therefore $\phi_1(x) = e^x$ for $x < 0$.

By differentiating ϕ_2 at $x < 0$, we find

$$\phi_2'(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ixk} \frac{ik}{(1 - ik)^2} dk = -\phi_1(x) + \phi_2(x).$$

The general solution to an equation $\phi_2' = -e^x + \phi_2$ is $\phi_2(x) = -xe^x + Ce^x$, but $\phi_2 \in \overset{o}{H}{}^l(R_-)$, $\forall\, l < 3/2$, is of the class $C^1(R)$, and hence, equal to 0 at $x = 0$. Thus, we obtain $\phi_2(x) = -xe^x$, $\forall\, x < 0$.

3) Now we calculate coefficients in (43):

$$f^\epsilon(0) = d(h) = A^0_-(-i)^{-1} q U' e^h - A^0_-(0)^{-1} r_1 K,$$

$$(f^\epsilon)'(0) = (1 + \epsilon) A^0_-(-i) q U' e^h - \epsilon A^0_-(0)^{-1} r_1 K = (1 + \epsilon) d(h) + A^0_-(0)^{-1} r_1 K.$$

4) As $x \to -0$,

$$e^{-\epsilon x} \phi_1(x) d(h) = (1 + (1 - \epsilon)x) d(h) + o(x),$$

$$e^{-\epsilon x} \phi_2(x) = -x + o(x),$$

$$e^{-\epsilon x} \phi_1(x) - x(d(h) - (1 + \epsilon) d(h) - A^0_-(0)^{-1} r_1 K) =$$

$$= d(h) + xd(h)[(1 - \epsilon) - 1 + (1 + \epsilon)] + x A^0_-(0)^{-1} r_1 K + o(x) =$$

$$= d(h) + xd(h) + x A^0_-(0)^{-1} r_1 K + o(x).$$

5) By gathering 1) – 4), we obtain (28).

A4. Calculation of $A^0_-(-i)$ and $A^0_-(0)$. By noticing that $\Re \hat{p}$ is even and $\Im \hat{p}$ is odd, and using (26), we obtain

$$b_-(-i) = -\frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\ln(1 - e^{-r+i\alpha l} \hat{p}(l))}{-i - l} dl =$$

$$= -\frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{i-l}{l^2+1} \ln\left(((1 - e^{-r}\hat{p}(l)\cos(\alpha l))^2 + (e^{-r}\hat{p}(l)\sin(\alpha l))^2)^{1/2}\right) dl -$$

$$-\frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{i-l}{l^2+1} \left(-i \arctan \frac{e^{-r}\hat{p}(l)\sin(\alpha l)}{1 - e^{-r}\hat{p}(l)\cos(\alpha l)}\right) dl =$$

$$= \frac{1}{2\pi} \int_0^\infty \frac{\ln((1 - e^{-r}\hat{p}(l)\cos(\alpha l))^2 + (e^{-r}\hat{p}(l)\sin(\alpha l))^2)}{l^2+1} dl -$$

$$-\frac{1}{\pi} \int_0^\infty \frac{l}{l^2+1} \arctan \frac{e^{-r}\hat{p}(l)\sin(\alpha l)}{1 - e^{-r}\hat{p}(l)\cos(\alpha l)} dl,$$

and

$$b_-(0) = -i\frac{1}{2\pi} \lim_{\tau \to +0} \int_{-\infty}^{+\infty} \frac{\ln(1 - e^{-r+\alpha l}\hat{p}(l))}{-i\tau - l} dl =$$

$$= -i\frac{1}{2\pi} \lim_{\tau \to +0} \int_{-\infty}^{+\infty} \frac{(i\tau - l)\ln(1 - e^{-r+\alpha l}\hat{p}(l))}{\tau^2 + l^2} dl =$$

$$= \frac{1}{2\pi} \ln(1 - e^{-r}) \int_0^\infty \frac{dl}{l^2+1} - \frac{1}{\pi} \int_0^\infty l^{-1} \arctan \frac{\hat{p}(l)\sin(\alpha l)}{e^r - \hat{p}(l)\cos(\alpha l)} dl =$$

$$= \frac{1}{2} \ln(1 - e^{-r}) - \frac{1}{\pi} \int_0^\infty l^{-1} \arctan \frac{\hat{p}(l)\sin(\alpha l)}{e^r - \hat{p}(l)\cos(\alpha l)} dl.$$

Since $A_-^0 = \exp b_-^0$,

$$\frac{A_-^0(-i)}{A_-^0(0)} = \exp\{-\frac{1}{2}\ln(1 - e^{-r}) + I_1 - I_2\},$$

where I_1, I_2 are the same as in (31).